Exact solution approaches for the order batching, sequencing, and picker routing problem with release times

Technical report

PUBLISHED ONLINE WITHIN THE *Working paper series of the chair of Management Science, Operations- and Supply Chain Management of the University of Passau*

Catherine Lorenz
Chair of Management Science / Operations
and Supply Chain Management,
University of Passau,
Dr.-Hans-Kapfinger-Straße 12, 94032 Passau,
catherine.lorenz@uni-passau.de

Michel Gendreau
Department of Mathematics
and Industrial Engineering
and CIRRELT,
Polytechnique Montréal,
Montréal, Canada michel.gendreau@polymtl.ca

Alena Otto
Chair of Management Science / Operations
and Supply Chain Management,
University of Passau,
Dr.-Hans-Kapfinger-Straße 12, 94032 Passau,
alena.otto@uni-passau.de

August 8, 2024

1 Introduction

Warehouses are used for an intermediate storage of commodities and are indispensable in a number of industries, including logistics, manufacturing, and retail. With digitization, spreading express-deliveries in e-commerce, and (mass-)customization in manufacturing, many companies have to reconsider their warehousing operations as online, with *dynamically* arriving orders to retrieve commodities from the one hand, and a pressure for a *fast*, but cost-efficient processing of these orders, on the other hand. Therefore, the interest in *online policies* for warehousing operations is high in the optimization literature Pardo et al. (cf. 2023).

A central instrument in understanding and improving the performance of online policies is the *gap to optimality*. However, the concept of *optimality* is not so obvious in an online, dynamic setting, as it is in case of deterministic optimization problems. Indeed, the sequentially incoming information – orders with specific characteristics arriving at certain times – may take a wide range of values, whose 'true' distribution is essentially unknown in practice. A concept of *complete-information optimum (CIOPT)* evolved as a state-of-the-art instrument to estimate optimality gaps of online policies (see Fiat and Woeginger, 1998). $CIOPT(I)$ for instance I refers to the best possible objective value computed by an omniscient oracle, who has the complete information on the future – i.e., on the incoming orders in the context of the warehousing operations.

The concept of CIOPT is useful for two main reasons. First of all, it can be interpreted as the best possible feasible online policy, including any anticipatory policies (cf. Fiat and Woeginger, 1998). Secondly, finding CIOPT amounts to solving a *deterministic* optimization problem, since all the future information is immediately available 'with certainty'.

Even if we would know the true distribution of the future information, it is usually computationally easier to find CIOPT than to solve the resulting stochastic multistage optimization problem.

In this paper, we develop algorithms to compute CIOPT in the context of warehousing operations. We focus on picking activities, since they consume more than the half of operating costs (Chen et al., 2015; Marchet et al., 2015), and investigate a widespread *pick-while-sort* setup in *picker-to-parts* warehouses. Picker-to-parts warehouses, in which the picker travels around the warehouse to collect ordered items, constitute the majority of the warehouses (Napolitano, 2012; Vanheusden et al., 2023). In the pick-while-sort setup, the picker is equipped with a cart and can process several orders at a time to save on costs. The cart consists of *bins*, each bin can accommodate the items of one order. Figure 1 illustrates two alternative cart technologies: a manual pushcart and a robotic cart. The order picking operations in the described setting refer to *the online order batching, sequencing, and routing problem (OOBSRP)* in the classification of Pardo et al. (2023), which is one of the most studied problems in the warehousing literature. To compute CIOPT, we have to solve exactly the complete-information counterpart of OOBSRP, which is *the order batching, sequencing, and routing problem with release times (OBSRP-R)*. Observe that since the arrival times of dynamically arriving orders are perfectly known to the omniscient oracle, they can be interpreted as release times.

To the best of our knowledge, no exact or heuristic methods have been proposed for OBSRP-R. Not surprisingly, there is *very little understanding of the optimality gaps* of the developed online policies for the online problem – OOBSRP. The contribution of this paper is the following:

- We design a *dynamic program (DP)* for OOBSRP-R, which can flexibly accommodate different cart technologies (a manual pushcart and a robotic cart) as well as different objective functions, including the cost-oriented minimization of the *total completion time* of the order picking operations.
- The proposed DP utilizes tailored *dominance rules* that reduce computational times significantly.

In the following, Section 2 states OBSRP-R formally and Section 3 presents our exact solution approach.

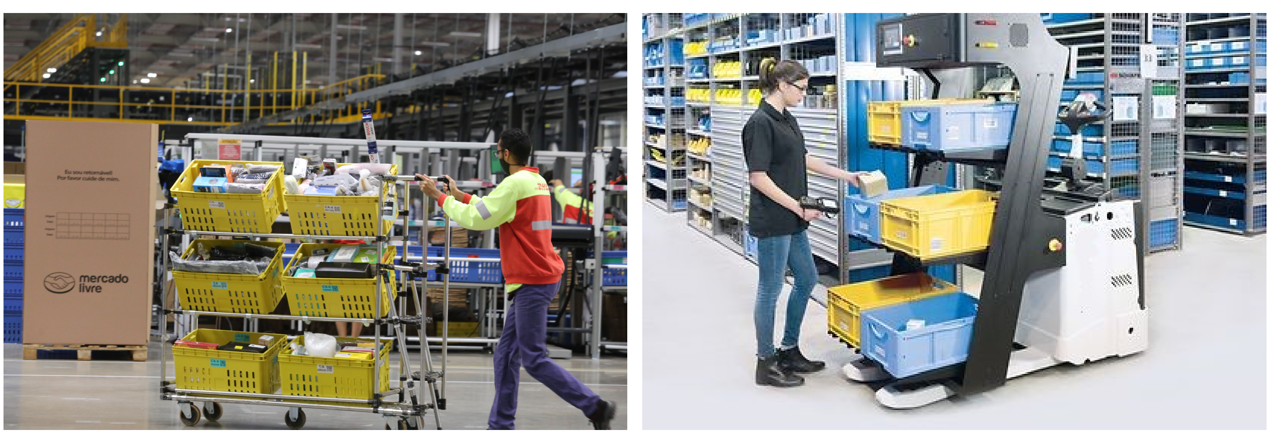
2 Problem statement

We define OBSRP-R with a pushcart and with a robotic cart, and discuss the underlying assumptions in Section 2.1. Afterward, Section 2.2 introduces some of the problem's properties.

Throughout the rest of the paper, we abbreviate $\{1, \ldots, n\}, \forall n \in \mathbb{N}$ simply as $[n]$.

2.1 The order batching, sequencing and routing problem with release times (OBSRP-R)

OBSRP-R describes picking operations of a single picker equipped with a cart (cf. Pardo et al., 2023). For simplicity, we call the working zone of this picker a *warehouse*. The warehouse is rectangular and of length L and width W. It consists of $a \geq 1$ vertical *aisles*, $b \geq 2$ horizontal *cross-aisles*, and a depot l_d which is positioned at some arbitrary location in the warehouse (see Figure 2). We dub the access point to the storage location, where the ordered item can be



Source: Governo do Estado de São Paulo. License.: CC BY 2.0 — Source: SSI Schäfer

(a) Pushcart (b) Robotic cart

Figure 1: Different types of picking carts in a picker-to-parts warehouse

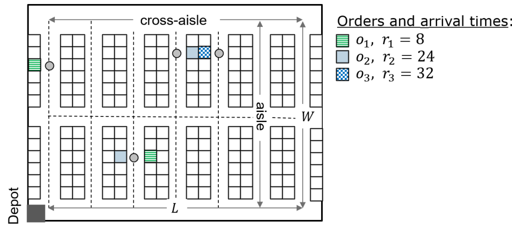


Figure 2: Illustration of a warehouse
Note. A warehouse of length L and width W in a rectilinear with three cross-aisles and seven aisles. The colored squares mark storage locations of the ordered items. The grey circles in the adjacent aisles mark the respective picking locations of these items.

retrieved, as the *picking location* of this item (see circles in Figure 2). Note that the picker can access the product items only from the aisles. Specifically, no items can be picked from the cross-aisles. The picker moves along aisles and cross-aisles, which results in a particular rectilinear distance metric $d()$.

The picker begins her operation at the depot and navigates through the warehouse at a constant *speed*, v. She has a cart with c bins, as depicted in Figure 1, where each bin is dedicated to a different order, so that items from the same order are grouped together in one bin. Consequently, up to c orders can be collected *simultaneously* (in one *batch*) to speed up the process. And we call c as the *batching capacity* in the following. Note that items from the same order cannot be split across different batches.

We examine two variants of OBSRP-R, each featuring a different cart technology: OBSRP-R with a *pushcart* and OBSRP-R with a *robotic cart*. The main difference between the carts is that the pushcart is moved by the picker, whereas the robotic cart can drive autonomously. Therefore, in case of a *pushcart*, the picker must return the cart with a completed batch to the depot for unloading; from there, she can pick an empty cart and start the next batch. Like this, a batch is completed *at the return to the depot*. In contrast, a *robotic cart* drives autonomously to the depot for unloading each time a batch is completed. Therefore, the picker can stay in the 'field' and move directly to the first picking location of the next batch, where a new, empty robotic cart is already waiting. Thus, in case of a robotic cart, the *completion time* of a batch corresponds to the time when the last item of this batch is picked.

Following the definition of OBSRP(-R) (cf. Pardo et al., 2023), we assume that each ordered item is associated with a *unique* picking location. This corresponds, for instance, to a hierarchical planning process, when the picking locations of the ordered items are determined first. Therefore, we use the terms *item* and a *picking location* interchangeably in the following.

An *instance* I of the OBSRP-R with a *a specific cart* has the following input (see Table 1):

- warehouse parameters, such as the number of aisles a, the number of cross-aisles b, length L, width W, distance metric $d()$;
- parameters of the picker and the cart, such as the picker's speed v, the batching capacity c, and constant pick time t^p to retrieve an item from its picking location;
- set of n^o orders. Each order $o_j, j \in [n^o]$:
  - represents a set of picking locations of the ordered items $o_j := \{s_j^1, ..., s_j^l\}, l \in \mathbb{N}$;
  - is associated with a release time $r_j \in \mathbb{R}^+$.

W.l.o.g., we assume $r_1 \leq r_2 \leq ... \leq r_{n^o}$. We also abuse the notation and denote the release time of item s in order o_j simply as $r(s) := r_j$. Each order can have a different number of picking locations. We denote the distance between some picking locations $s_j^i, s_{j'}^{i'}$, as $d(s_j^i, s_{j'}^{i'})$. Similarly, $d(l_d, s_j^i)$ refers to the distance between location s_j^i and the depot. We assume that the distances are metric, for instance, that the *triangular inequalities* hold: $\forall i, l, k \in S \cup \{l_d\} : d(i, k) \leq d(i, l) + d(l, k)$. The latter is true, for instance, if the distance is measured along a shortest path between the corresponding locations. For convenience, we denote the total number of items as $n^i := \sum_{j=1}^{n^o} |o_j|$ and the set of items as $S := \bigcup_{j \in [n^o]} \{o_j\}$. The retrieval of each item requires fixed *pick time* t^p.

Let denote the *completion time* of an ordered item $s \in S$ in some solution σ as $C(s, \sigma)$, which is the time when the item is placed in the cart. In the following, we abuse the notation and drop the reference to σ, if it is clear from the context.

A *feasible solution* σ of OBSRP-R for a given instance I consists of:

- the visiting sequence of picking locations π, which is constructed as $\pi := (\pi^{B_1}, \pi^{B_2}, \ldots, \pi^{B_f})$, $|\pi| = n^i$, based on the following components:
  - an ordered sequence of batches $\pi^{\text{batches}} = (B_1, B_2, \ldots, B_f)$, such that the batches form a mutually disjoint partition of the orders: $\{o_1, ..., o_{n^o}\} = B_1 \cup B_2 \cup \ldots \cup B_f, B_l \cap B_k = \emptyset \; \forall k, l \in \{1, ..., f\}$; each batch contains at most c orders. The number of batches $f \in \mathbb{N}$ is a decision variable.
  - for each batch B_l, a permutation of the picking locations of all the included orders π^{B_l}.
- the picking *schedule* consisting of a completion time $C(s)$ for each item $s \in S$, such that the routing requirements for the specific type of cart are respected, and no item s is picked before its release time $r(s)$. Specifically, the values $C(s), s \in S$ obey equations (5)-(6) in case of a *pushcart* and (3)-(4) in case of a *robotic cart*, respectively, as stated below.

The objective function is to minimize the *total completion time* of picking. This cost-centered objective revolves around the working time of the picker and is defined as follows:

$$Minimize_\sigma \; z(\sigma) = \max_{s \in S} C(s) \qquad \text{in the case of a } \textit{robotic cart} \tag{1}$$

$$Minimize_\sigma \; z(\sigma) = \max_{s \in S} (C(s) + \frac{1}{v} \cdot d(s, l_d)) \qquad \text{in the case of a } \textit{pushcart} \tag{2}$$

In Objective (2), the picker has to return the cart to the depot for unloading at the end of picking operations.

Without losing optimality, we assume that the picker collects each item at the *earliest possible time* (which accounts for the release times of the orders). Therefore, given the sequence of picking locations π and the respective sequence of batches π^{batches}, we can compute the schedule in the unique way. For convenience, let denote the i^{th} item in π as $\pi[i]$. In case of a *robotic cart*:

$$C(\pi[1]) = \max\{\frac{1}{v} \cdot d(l_d, \pi[1]), r(\pi[1])\} + t^p \tag{3}$$

$$C(\pi[i+1]) = \max\{C(\pi[i]) + \frac{1}{v} \cdot d(\pi[i], \pi[i+1]), r(\pi[i+1])\} + t^p \qquad \forall i \in [n^i - 1] \tag{4}$$

In case of a *pushcart*, we have to return the cart to the depot between two subsequent batches, therefore:

$$C(\pi[1]) = \max\{\frac{1}{v} \cdot d(l_d, \pi[1]), r(\pi[1])\} + t^p \tag{5}$$

$$C(\pi[i+1]) = \begin{cases} \max\{C(\pi[i]) + \frac{1}{v} \cdot d(\pi[i], \pi[i+1]), r(\pi[i+1])\} + t^p & \text{if } \pi[i], \pi[i+1] \text{ belong to the same batch} \\ \max\{C(\pi[i]) + \frac{1}{v} \cdot d(\pi[i], l_d) + \frac{1}{v} \cdot d(l_d, \pi[i+1]), r(\pi[i+1])\} + t^p & \text{if } \pi[i], \pi[i+1] \text{ belong to distinct batches} \end{cases}$$
$$\forall i \in [n^i - 1] \tag{6}$$

2.2 Selected dominance relations in OBSRP-R with a robotic cart

In the designed exact solution algorithm, we reduce the solution space for an OBSRP-R instance I with by excluding weakly dominated solutions. A feasible solution $\hat{\sigma}$ is *weakly dominated* by some other feasible solution $\tilde{\sigma}$ if $z(\tilde{\sigma}) \leq z(\hat{\sigma})$.

Proposition 1 of this section formulates some dominance relations for OBSRP-R with a *robotic cart*. We state several further dominance relations for OBSRP-R with both alternative cart technologies in Section 3.3, after the description of the dynamic program (Sections 3.1-3.2), because we will rely on the notation of Section 3.1to explain, how these rules are integrated into the solution procedure.

The message of Proposition 1 is very simple: If possible, it is always worth closing a batch and getting an empty robotic cart. In other words, as soon as all items of the already commenced orders of the current batch have been picked, the picker should close the batch and start a new one with a timely arrived empty robotic cart, since the resulting total completion time will *ceteris paribus* not increase.

Proposition 1. *Consider any feasible solution $\tilde{\sigma}$ with a visiting sequence of picking locations $\tilde{\pi}$ and the respective sequence of batches $\tilde{\pi}^{batches}$, $|\tilde{\pi}^{batches}| = \tilde{f} \in \mathbb{N}$ that schedules two batches $\tilde{B}_l, \tilde{B}_{l+1}, l \in [\tilde{f} - 1]$, with the following property subsequently:*

$$|\tilde{B}_l| + |\tilde{B}_{l+1}| \leq c \tag{7}$$

Then, $\tilde{\sigma}$ weakly dominates the following feasible solution $\hat{\sigma}$:

Table 1: Notation

Parameters of an OBSRP-R instance I	
L	Length of the warehouse
W	Width of the warehouse
a	The number of aisles
b	The number of cross-aisles
l_d	The position of the depot in the warehouse
$d()$	Distance metric for locations in the warehouse
v	Picker speed
t^p	Pick time: the time to retrieve one item from its storage location and place it into the cart
c	Batching capacity: the maximum number of orders in a batch
n^o	The number of orders
$o_j = \{s_j^1, ...s_j^l\}$	The set of items requested by the j^{th} order, $j \in [n^o], l \in \mathbb{N}$
r_j	Release time of order o_j, $j \in [n^o]$, $r_1 \leq ... \leq r_n$
$r(s)$	Release time of an item s, equals the release time of its respective order, $r(s) := r_j$ if $s \in o_j$
n^i	Total number of items, $n^i := \sum_{j=1}^{n^o} \lvert o_j \rvert$
S	The set of all ordered items of the instance, $S := \cup_{j \in [n^o]} o_j$
$C(\sigma, s)$	Completion time of item $s \in S$ in solution a solution σ of instance I (σ can be dropped if clear from the context)
Notation used in the DP-approach	
O	The sequence of orders sorted with respect to their release times
$\Theta_k = (s, m^o, S^{\text{batch}}, O^{\text{pend}})$	A state at stage $k \in [n^i + 1] \cup \{0\}$; for $k \in [n^i]$, k items have been picked at this state; Θ_0 is the initial state; Θ_{n^i+1} is the terminal state; $s \in S \cup l_d$ denotes the picker's current position; $m^o \in [c] \cup \{0\}$ counts the number of orders currently assigned to the open batch; set $S^{\text{batch}} \subseteq S$ accommodates the items of the orders from the current batch which have not been picked yet; $O^{\text{pend}} \vert O$ is the sequence of pending orders
$X(\Theta_k)$	Set of feasible transitions from state Θ_k
$f(\Theta_k, x_k)$	Transition function for state Θ_k at stage k and the feasible transition $x_k \in X(\Theta_k)$; the image is a state at the next stage Θ_{k+1}
$g(\Theta_k, x_k)$	Cost of a feasible transition $x_k \in X(\Theta_k)$ at state Θ_k, corresponds to the traveled distance
$\Omega(\Theta_k)$	Value of state S_k; the earliest possible time to reach state S_k

- *$\pi(\hat{\sigma}) = \tilde{\pi}$: The visiting sequence of picking locations in $\hat{\sigma}$ is identical to that of $\tilde{\sigma}$*
- *The sequence of batches $\pi^{batches}(\hat{\sigma})$ results from the sequence of batches $\tilde{\pi}^{batches}$ by replacing $\tilde{B}_l, \tilde{B}_{l+1}$ with one single batch $B_l = \tilde{B}_l \cup \tilde{B}_{l+1}$.*

Proof. By construction, $\hat{\sigma}$ is a feasible solution, for instance, $\pi^{\text{batches}}(\hat{\sigma})$ represents a mutually disjoint partition of the orders into batches and each batch contains at most c orders. What remains to show, is that $z(\tilde{\sigma}) \leq z(\hat{\sigma})$.

Recall that both solutions have the same visiting sequence of the picking locations $\pi(\hat{\sigma}) = \tilde{\pi}$. Let compute the schedule of both solutions $\tilde{\sigma}$ and $\hat{\sigma}$ as described in (3) and (4). It is straightforward to see that $C(\tilde{\pi}[i]) = C(\pi(\hat{\sigma})[i]), \forall i \in [n^i]$. By the definition of the objective function in (1), it follows that $z(\tilde{\sigma}) \leq z(\hat{\sigma})$.

□

3 Dynamic program

In this section, we describe the developed exact solution algorithm – *the dynamic program (DP)*. Sections 3.1 describes the state graph and Sections 3.2 states the Bellman equation for OBSRP-R with both alternative cart technologies, respectively. We highlight the DP specifics in the cases of a pushcart and of a robotic cart directly in the text. Afterward, Section 3.3 discusses several dominance relations, which we use to speed up the developed exact solution approach.

3.1 State graph

Recall that we sort the orders non-decreasingly with respect to their release times. Let denote this sorted sequence of orders as O and, for convenience, abbreviate the expression *"O' is an ordered subsequence of O"* as $O'|O$.

We reinterpret OBSRP-R as a sequential optimization problem with $k \in \{0, 1, ..., n^i, n^i + 1\}$ stages. At stage $k \leq n^i$, the picker has picked k items and, given the current *state* (such as her location, available bins in the cart etc.), faces a *subproblem* to optimally pick the remaining items of the remaining orders. Her immediate *decision* at stage k is about the next item to collect. Stage $n^i + 1$ is reserved for the terminal state as explained below. We depict the resulting sequential problem as a *state graph*. The nodes of this graph are called *states*. The directed edges depict *transitions* between the states of the subsequent stages and are associated with *decisions*.

States. We define state Θ_k at stage $k \in [n^i]$ as a tuple of the following four variables:

$$\Theta_k \in \{(s, m^o, S^{\text{batch}}, O^{\text{pend}}) | s \in S, m^o \in \{0, 1, \ldots, c\}, S^{\text{batch}} \subseteq S, O^{\text{pend}} | O\} \tag{8}$$

Variable $s \in S$ denotes the picker's current position, which coincides with the picking location of the last picked item. Integer $m^o \in \{0, 1, \ldots, c\}$ counts the number of orders currently assigned to the open batch. The *set* $S^{\text{batch}} \subseteq S$ accommodates the items of the orders from the current batch, which have not been picked yet. Sequence $O^{\text{pend}}|O$ is the *sequence* of pending orders, no items of which have been picked so far.

In the *initial* state at stage $k = 0$, the picker is in the depot with an empty cart: $\Theta_0 = (l_d, 0, \{\}, O)$. The last stage $k = n^i + 1$ consists of one state, which we dub *terminal state*. In the case of a *robot*, it is a dummy state. In the case of a *pushcart*, the terminal state describes the picker's return to the depot with all the orders processed: $S_{n^i+1} = (l_d, 0, \{\}, ())$.

States of type $\Theta_k = (s, 0, \{\}, O^{\text{pend}})$ with $S^{\text{batch}} = \emptyset$ and $m^o = 0$ mark a completed batch in the designed state graph, s is the last picked item of this batch. In OBSRP-R with a *pushcart*, the picker will accompany the completed batch to the depot. In OBSRP-R with a *robotic cart*, the picker proceeds directly to the first item of the subsequent batch, since the loaded robotic cart returns to the depot autonomously. We call these states as *batch-completion states* in the following.

Transitions. We denote a set of feasible decisions, or *feasible transitions*, from state Θ_k at stage $k \leq n^i$ as $X(\Theta_k)$. The *transition function* $f(\Theta_k, x_k) = \Theta_{k+1}$ describes the next state after the execution of the decision $x_k \in X(\Theta_k)$ at state Θ_k, and we denote the costs of the corresponding transition as $g(\Theta_k, x_k)$. The set of feasible transitions and the associated costs differ depending on the cart technology – a pushcart or a robotic cart –, as we explain below.

At stage $k = n^i$, there is one only feasible transition in each state Θ_{n^i}, which is to move to the terminal state. Transition costs in case of a *pushcart* equal the walking distance from the last visited picking location to the depot, $g(\Theta_{n^i}, x_{n^i}) := d(s, l_d)$, since the picker returns the cart to the depot. In case of a *robotic cart*, $g(\Theta_{n^i}, x_{n^i}) := 0$.

At the remaining stages $k = \{0, 1, \ldots, n^i - 1\}$, transitions $x_k \in X(\Theta_k)$ refer to the selection of the next picking item s_{k+1}, and, potentially to the decision of extending- or completing the currently open batch. In case of a *robotic cart*, transition costs always equal $g(\Theta_k, x_k) = d(s_k, s_{k+1})$, which corresponds to the distance between the current location s_k and the next picking location s_{k+1} of the picker. In the case of a *pushcart*, transition costs differ between the batch-completion states and the remaining states, because the picker has to return the cart to the depot after the batch is completed. Therefore, $g(\Theta_k, x_k) = d(s_k, l_d) + d(l_d, s_{k+1})$ if Θ_k is a batch-completion state, and $g(\Theta_k, x_k) = d(s_k, s_{k+1})$ else.

Table 2 describes feasible transitions for OBSRP-R with a *pushart*. Lines 1 and 2 describe transitions from a batch-completion state. These transitions initiate a new batch. Note that, if the next batch starts by picking an item from a *single-item* order, the picker has two alternatives: Either to pick this item as part of a larger batch (line 1) or to limit the batch to this one order only (line 2). In lines 3 - 7 and 10, the currently open batch is extended by a pending order. Thereby, in lines 3 to 6, there are still empty bins in the current batch ($m^o < c$), but all the items of the already assigned orders have been collected. If an item s_{k+1} from a *single-item* order is selected to be picked next, two alternatives must be distinguished: either the batch is completed by this order (i.e., the next state Θ_{k+1} is a batch-completion state, see lines 4 and 6), or further orders will be assigned to the current batch (lines 3 and 5). In lines 8-9 and 11-13, at least one item from a commenced order remains in S^{batch}. Similarly to previous transitions, if one item remains in S^{batch} and the batching capacity has not been depleted ($m^o \leq c - 1$) (lines 11 and 12), the picker can either close the batch after picking this remaining item and move to a batch-completion state (see line 12), or proceed by extending further the current batch (line 11).

Similarly, Table 3 defines feasible transitions for OBSRP-R with a *robotic cart*. The construction of the transitions follows the same logic as in the case of a *pushcart*. Observe, however, that we prohibit weakly dominated solutions described in Proposition 1 and force the picker to close the current batch and initiate a new one as soon as she has collected all the items of the currently assigned orders.

The next Proposition claims a one-to-one correspondence between the paths in the constructed state graph and the solution space of OBSRP-R.

Proposition 2. *For a particular OBSRP-R instance with a pushcart, each path from the initial state to the terminal state in the constructed state graph corresponds to exactly one feasible solution for this instance and vice-versa.*

In the case of a robotic cart, each path from the initial state to the terminal state in the constructed state graph corresponds to exactly one feasible solution in the reduced solution space by Proposition 1, and vice-versa.

Proof. For both cart technologies - *pushcart* and *robotic cart* - the first entry in each state of the state graph at stage $k \in [n^i]$ represents one picked item, thus, the sequence of these entries in a path of states corresponds to exactly one

Table 2: OBSRP-R with a *pushcart*: Feasible transitions at stages $k = \{0, 1, \ldots, n^i - 1\}$

	Current state Θ_k	Transition $x_k \in X(\Theta_k)$	New state $\Theta_{k+1} = f(\Theta_k, x_k)$
1	$(s_k, 0, \{\}, O^{\text{pend}})$	$\forall s_{k+1} \in o_j, o_j \in O^{\text{pend}}$	$(s_{k+1}, 1, o_j \setminus \{s_{k+1}\}, O^{\text{pend}} \setminus o_j)$
2	-//-	$\forall s_{k+1} \in o_j, o_j \in O^{\text{pend}}, \lvert o_j \rvert = 1$	$(s_{k+1}, 0, \{\}, S_k^{\text{pend}} \setminus o_j)$
3	$(s_k, m^o, \{\}, O^{\text{pend}}), 0 < m^o < c-1$	$\forall s_{k+1} \in o_j, o_j \in O^{\text{pend}}$	$(s_{k+1}, m^o+1, o_j \setminus s_{k+1}, O^{\text{pend}} \setminus o_j)$
4	-//-	$\forall s_{k+1} \in o_j, o_j \in O^{\text{pend}}, \lvert o_j \rvert = 1$	$(s_{k+1}, 0, \{\}, O^{\text{pend}} \setminus o_j)$
5	$(s_k, m^o, \{\}, O^{\text{pend}}), m^o = c-1$	$\forall s_{k+1} \in o_j, o_j \in O^{\text{pend}}, \lvert o_j \rvert > 1$	$(s_{k+1}, m^o+1, o_j \setminus s_{k+1}, O^{\text{pend}} \setminus o_j)$
6	-//-	$\forall s_{k+1} \in o_j, o_j \in O^{\text{pend}}, \lvert o_j \rvert = 1$	$(s_{k+1}, 0, \{\}, O^{\text{pend}} \setminus o_j)$
7	$(s_k, m^o, S^{\text{batch}}, O^{\text{pend}}), 0 < m^o \le c-1, \lvert S^{\text{batch}} \rvert > 1$	$\forall s_{k+1} \in o_j, o_j \in O^{\text{pend}}$	$(s_{k+1}, m^o+1, S^{\text{batch}} \cup o_j \setminus s_{k+1}, O^{\text{pend}} \setminus o_j)$
8	-//-	$\forall s_{k+1} \in S^{\text{batch}}$	$(s_{k+1}, m^o, S^{\text{batch}} \setminus \{s_{k+1}\}, O^{\text{pend}})$
9	$(s_k, m^o, S^{\text{batch}}, O^{\text{pend}}), m^o = c, \lvert S^{\text{batch}} \rvert > 1$	$\forall s_{k+1} \in S^{\text{batch}}$	$(s_{k+1}, m^o, S^{\text{batch}} \setminus \{s_{k+1}\}, O^{\text{pend}})$
10	$(s_k, m^o, S^{\text{batch}}, O^{\text{pend}}), 0 < m^o \le c-1, \lvert S^{\text{batch}} \rvert = 1$	$\forall s_{k+1} \in o_j, o_j \in O^{\text{pend}}$	$(s_{k+1}, m^o+1, S^{\text{batch}} \cup o_j \setminus s_{k+1}, O^{\text{pend}} \setminus o_j)$
11	-//-	$\forall s_{k+1} \in S^{\text{batch}}$	$(s_{k+1}, m^o, \{\}, O^{\text{pend}})$
12	-//-	$\forall s_{k+1} \in S^{\text{batch}}$	$(s_{k+1}, 0, \{\}, O^{\text{pend}})$
13	$(s_k, m^o, S^{\text{batch}}, O^{\text{pend}}), m^o = c, \lvert S^{\text{batch}} \rvert = 1$	$\forall s_{k+1} \in S^{\text{batch}}$	$(s_{k+1}, 0, \{\}, O^{\text{pend}})$

Note. Lines 1 and 2 refer to a transition from a batch-completion state with transition costs of $g(\Theta_k, x_k) = d(s_k, l_d) + d(l_d, s_{k+1})$.
The costs of the remaining transitions equal $g(\Theta_k, x_k) = d(s_k, s_{k+1})$.

Table 3: OBSRP-R with a *robotic cart*: Feasible transitions at stages $k = \{0, 1, \ldots, n^i - 1\}$

	Current state Θ_k	Transition $x_k \in X(\Theta_k)$	New state $\Theta_{k+1} = f(\Theta_k, x_k)$
1	$(s_k, 0, \{\}, O^{\text{pend}})$	$\forall s_{k+1} \in o_j, o_j \in O^{\text{pend}}, \lvert o_j \rvert > 1$	$(s_{k+1}, 1, o_j \setminus \{s_{k+1}\}, O^{\text{pend}} \setminus o_j)$
2	-//-	$\forall s_{k+1} \in o_j, o_j \in O^{\text{pend}}, \lvert o_j \rvert = 1$	$(s_{k+1}, 0, \{\}, S_k^{\text{pend}} \setminus o_j)$
3	$(s_k, m^o, S^{\text{batch}}, O^{\text{pend}}), 0 < m^o \le c-1, \lvert S^{\text{batch}} \rvert > 1$	$\forall s_{k+1} \in S^{\text{batch}}$	$(s_{k+1}, m^o, S^{\text{batch}} \setminus \{s_{k+1}\}, O^{\text{pend}})$
4	-//-	$\forall s_{k+1} \in o_j, o_j \in O^{\text{pend}}$	$(s_{k+1}, m^o+1, S^{\text{batch}} \cup o_j \setminus s_{k+1}, O^{\text{pend}} \setminus o_j)$
5	$(s_k, m^o, S^{\text{batch}}, O^{\text{pend}}), 0 < m^o \le c-1, \lvert S^{\text{batch}} \rvert = 1$	$\forall s_{k+1} \in S^{\text{batch}}$	$(s_{k+1}, 0, \{\}, O^{\text{pend}})$
6	-//-	$\forall s_{k+1} \in o_j, o_j \in O^{\text{pend}}$	$(s_{k+1}, m^o+1, S^{\text{batch}} \cup o_j \setminus s_{k+1}, O^{\text{pend}} \setminus o_j)$
7	$(s_k, m^o, S^{\text{batch}}, O^{\text{pend}}), m^o = c, \lvert S^{\text{batch}} \rvert > 1$	$\forall s_{k+1} \in S^{\text{batch}}$	$(s_{k+1}, m^o, S^{\text{batch}} \setminus \{s_{k+1}\}, O^{\text{pend}})$
8	$(s_k, m^o, S^{\text{batch}}, O^{\text{pend}}), m^o = c, \lvert S^{\text{batch}} \rvert = 1$	$\forall s_{k+1} \in S^{\text{batch}}$	$(s_{k+1}, 0, \{\}, O^{\text{pend}})$

Note. The costs of *all* transitions equal $g(\Theta_k, x_k) = d(s_k, s_{k+1})$.

visiting sequence of picking locations π. A partitioning of this sequence into batches can be derived by closing one batch after each visit of a batch-completion state and opening a new batch with the picking location of the following state. By construction, the resulting batches form a disjoint partition of all n^o orders of the instance, and each batch contains at most c orders. In the case of a *robotic cart*, lines 2, 5, and 8 of Table 3 enforce a batch-completion state in each path through the state graph each time all the items from the commenced orders of the batch are picked. Thus, none of the feasible solutions that correspond to a path in the state graph is weakly dominated in the sense of Proposition 1.

The other way around, each feasible solution, which is non-dominated in the sense of Proposition 1 in the case of a *robotic cart*, translates to exactly one path of states connected by transitions from Tables 2 and 3 in the case of a *pushcart* and *robotic cart*, respectively, by construction. □

3.2 Bellman equations

Let define the *value* $\Omega(\Theta_k)$ of state Θ_k at any stage k as the earliest possible time to reach this state starting from the initial state in the state graph. We set the value of the initial state Θ_0 to be 0: $\Omega(\Theta_0) := 0$. By this definition, the *optimal* objective value for OBSRP-R instance I is the value of the terminal state Θ_{n^i+1} in the respective state graph.

We compute state values in a forward-induction manner starting from initial state $\Omega(\Theta_0)$ by using the following Bellman equations, which are valid for OBSRP-R with both cart technologies – a pushcart and a robotic cart:

$$\Omega(\Theta_{k+1}) = \begin{cases} \min_{(\Theta_k, x_k) \in f^{-1}(\Theta_{k+1})} \{\max\{r(s), \Omega(\Theta_k) + \frac{1}{v} \cdot g(\Theta_k, x_k)\} + t^p\} & \forall k \in \{0, ..., n^i - 1\} \\ \min_{(\Theta_k, x_k) \in f^{-1}(\Theta_{k+1})} \{\Omega(\Theta_k) + \frac{1}{v} \cdot g(\Theta_k, x_k)\} & \text{for } k = n^i \end{cases} \tag{9}$$

Function $f^{-1}(\Theta_{k+1}) = \{(\Theta_k, x_k) \mid \Theta_k$ is a state at stage $k, x_k \in X(\Theta_k)$, and $f(\Theta_k, x_k) = \Theta_{k+1}\}$ is a reverse transition function and represents the set of all feasible transitions to reach Θ_{k+1} from some state Θ_k at stage k (see Tables 2 and 3). Release time $r(s)$ refers to the current location of the picker in the respective state $\Theta_k = (s, m^o, S^{\text{batch}}, O^{\text{pend}})$.

By construction, the values of states Θ_k at stages $k \in [n^i]$ correspond to the earliest possible completion time $C(s)$ of picking the last item $s \in S$ of this state among all the possible feasible partial solutions described by the state (cf. equations (3)-(6) in Section 2). Similarly, the *optimal* objective value, i.e. the minimal total completion time for OBSRP-R instance I, is the value of the terminal state Θ_{n^i+1} in the respective state graph, see objectives (1) and (2).

3.3 Dominance rules for transitions in the DP formulation

In this section, we show how some transitions in the state graph of the dynamic program can be omitted, and, thus, the complexity of the DP can be reduced, without compromising its optimality. We say that a *transition* in the state graph is *weakly dominated*, if, for every path from the initial to the terminal state involving this transition, the corresponding feasible solution of the instance (see Proposition 3.1) is weakly dominated by some other feasible solution whose path from the initial to the terminal state that does not involve this transition.

Propositions 3, 4 and 5, define dominance rules for transitions, which prohibit the picker selecting items s for the next pick, whose release time $r(s)$ lies too far into the future. The rules leverage the properties of the warehouse metric $d()$.

The following proposition holds both for OBSRP-R with a *pushcart* and for OBSRP-R with a *robotic cart*.

Proposition 3. *Consider a current state $\Theta_k = (s_k, m^o, S^{batch}, O^{pend})$ with $|S^{batch}| \geq 1$. Then any transition $x_k \in X(\Theta_k)$ that visits next picking location $s_{k+1} \in o_j, o_j \in O^{pend}$ is weakly dominated, if*

$$r_j \geq \Omega(\Theta_k) + \frac{1}{v} \cdot 2(W + L) + t^p \tag{10}$$

Proof. See Appendix 4.1. □

Note that the expression $\frac{1}{v} \cdot 2(W + L) + t^p$ in (10) is an upper bound for the time needed by the picker to start in any location of the warehouse, move to the picking location of any ordered item, complete the picking of the latter and move to any other location in the warehouse. In other words, Proposition 3 prohibits selecting item s_{k+1} with a late release time, if there is another item in a commenced order of the current batch ($|S^{\text{batch}}| \geq 1$) that can be picked first.

The next proposition holds in the case of a *pushcart*.

Proposition 4. *Consider a current state $\Theta_k = (s_k, m^o, \{\}, O^{pend})$ with $m^o > 0$. Then any transition $x_k \in X(\Theta_k)$ that visits next picking location $s_{k+1} \in o_j, o_j \in O^{pend}$ is weakly dominated for OBSRP-R with a pushcart, if*

$$r_j \geq \Omega(\Theta_k) + \frac{1}{v} \cdot 2(W + L) \tag{11}$$

Proof. See Appendix 4.2. □

In this case, the expression $\frac{1}{v} \cdot 2(W + L)$ in (11) is an upper bound for the time needed by the picker to start in any location of the warehouse, visit the depot, and move to any other location in the warehouse. In other words, Proposition 4 requires completing the current batch before moving to item s_{k+1} with a late enough release time, if all the items of the commenced orders in the current batch have been picked.

Finally, Proposition 5 describes dominance relations for feasible transitions from batch-completion states for both alternative cart technologies. It prevents starting a new batch with a pending order $o_{\bar{j}} \in O^{\text{pend}}$ that has a late release time, if we can complete another pending order $o_{\underline{j}}$ in a one-order batch first.

Proposition 5. *Consider a current state $\Theta_k = (s_k, 0, \{\}, O^{pend})$. Then, any transition $x_k \in X(\Theta_k)$ that visits next picking location $s_{k+1} \in o_{\bar{j}}, o_{\bar{j}} \in O^{pend}$ is dominated, if there exists $o_{\underline{j}} \in O^{pend}$ such that*

$$r_{\bar{j}} \geq \max\{\Omega(\Theta_k); r_{\underline{j}}\} + \frac{1}{v} \cdot UB(o_{\underline{j}}) + |o_{\underline{j}}| \cdot t^p \qquad \textit{in case of a robotic cart} \tag{12}$$

$$r_{\bar{j}} \geq \max\{\Omega(\Theta_k) + \frac{1}{v} \cdot d(s_k, l_d); r_{\underline{j}}\} + \frac{1}{v} \cdot (UB(o_{\underline{j}}) + W + L) + |o_{\underline{j}}| \cdot t^p \qquad \textit{in case of a pushcart} \tag{13}$$

Thereby, UB is an upper bound for the minimum travel distance to visit all picking locations in $o_{\underline{j}}$, starting and ending in fixed, given points in the warehouse.

Proof. See Appendix 4.3. □

Appendix 4.4 suggests $UB(o_j) := (n^{\text{aisles}}(o_j) + 1)W + 2L$, for any order o_j, where $n^{\text{aisles}}(o_j)$ equals the number of aisles that contain the picking locations of o_j. The summand $UB(o_{\underline{j}}) + W + L$ in (13) refers to the minimum walking distance to complete $(o_{\underline{j}})$ in a closed tour from the depot, and then move to an arbitrary next picking location in the warehouse.

In the designed algorithm, we apply the dominance relations of Propositions 3-5 as follows. In each state Θ_k, transitions $x_k \in X(\Theta_k)$, which select the next item s_{k+1} to pick, are considered in the non-decreasing order of the release times of these items. By the nature of the dominance rules (10)-(13), if one transition is weakly dominated, then all the following transitions from Θ_k are weakly dominated as well. Algorithm 1 describes the implementation of the dominance rules from Propositions 3 and 5 in the case of a *robotic cart*. The implementation of the dominance rules in the case of a *pushcart* follows the similar lines.

Algorithm 1: Construction of non-dominated transitions from state $\Theta_k = (s_k, m^o, S^{\text{batch}}, O^{\text{pend}})$

1 Let $o_{\underline{j}} \in O^{\text{pend}}$ be the first order with the minimum release time in O^{pend}.
2 **for** *all* $s_{k+1} \in S^{batch}$ *in arbitrary order* **do**
3   perform applicable transition(s) of Table 3
4 **end**
5 **for** *all* $o_j \in O^{pend}$ *in the given order of* O^{pend} **do**
6   **if** *(* $|S^{batch}| \geq 1$ *and inequality (10) is True)* **or** *(* $m_0 = 0$ *and inequality (12) is True)* **then**
7   break and go to line 13;
8   **else**
9   **for** *all* $s_{k+1} \in o_j$ *in arbitrary order* **do**
10   perform applicable transition(s) of Table 3
11   **end**
12   **end**
13 **end**

Note. Case of a robotic cart

References

T.-L. Chen, C.-Y. Cheng, Y.-Y. Chen, and L.-K. Chan. An efficient hybrid algorithm for integrated order batching, sequencing and routing problem. *International Journal of Production Economics*, 159:158–167, 2015.

A. Fiat and G. J. Woeginger. *Online Algorithms*, volume 1442. Springer: Berlin, 1998.

G. Marchet, M. Melacini, and S. Perotti. Investigating order picking system adoption: a case-study-based approach. *International Journal of Logistics Research and Applications*, 18(1):82–98, 2015.

M. Napolitano. 2012 warehouse/DC operations survey: mixed signals. *Logistics management (Highlands Ranch, Colo.: 2002)*, 51(11), 2012.

E. G. Pardo, S. Gil-Borrás, A. Alonso-Ayuso, and A. Duarte. Order batching problems: Taxonomy and literature review. *European Journal of Operational Research*, 313(1):1–24, 2023.

S. Vanheusden, T. van Gils, K. Ramaekers, T. Cornelissens, and A. Caris. Practical factors in order picking planning: state-of-the-art classification and review. *International Journal of Production Research*, 61(6):2032–2056, 2023.

4 Appendix

4.1 Proof of Proposition 3

In this section, we consider an arbitrary instance I for OBSRP-R and show that the dominance rule claimed in Proposition 3 is valid for both cart technologies - a *pushcart* and a *robotic cart*.

We use Lemma 1 to define a set D_1 of feasible solutions for OBSRP-R instance I that are weakly dominated by other feasible solutions. In the second step, Lemma 2 states that every path from the initial state to the terminal state in the constructed state graph, which involves a transition $x_k, k \in [n^i - 1]$ described by Proposition 3, is associated to a solution which is either in set D_1, or, which is weakly dominated by a solution in set D_1. Thus, by definition, transition x_k is *weakly dominated*.

Lemma 1. *Consider any feasible solution $\hat{\sigma}$ with a visiting sequence of picking locations $\hat{\pi}$ and the respective sequence of batches $\hat{\pi}^{batches}$ such that:*

- *Some item $\hat{\pi}[i+1]$ has a late release date:*

$$r(\hat{\pi}[i+1]) \geq C(\hat{\pi}[i]) + \frac{1}{v} \cdot 2(W+L) + t^p \tag{14}$$

- *There exists some order $o_{\hat{j}}$, some items of which are picked before item $\hat{\pi}[i+1]$ and some items of which are picked after item $\hat{\pi}[i+1]$. Let denote an item of $o_{\hat{j}}$, which is picked after item $\hat{\pi}[i+1]$ in $\hat{\sigma}$ as $s^\dagger$.*

Let decompose the visiting sequence of $\hat{\sigma}$ as follows: $\hat{\pi} = (\hat{\pi}^I, \hat{\pi}[i+1], \hat{\pi}^{II})$, i.e. $\hat{\pi}^I$ and $\hat{\pi}^{II}$ denote the subsequences before and after the visit of location $\hat{\pi}[i+1]$, respectively.

Then the following solution $\tilde{\sigma}$ weakly dominates solution $\hat{\sigma}$:

- $\pi^{batches}(\tilde{\sigma}) = \hat{\pi}^{batches}$
- $\pi(\tilde{\sigma}) = (\hat{\pi}^I, s^\dagger, \hat{\pi}[i+1], \hat{\pi}^{II} \setminus s^\dagger)$, *where $\hat{\pi}^{II} \setminus s^\dagger$ is the sequence $\hat{\pi}^{II}$ after removing item $s^\dagger$.*
  In other words, solution $\tilde{\sigma}$ collects item $s^\dagger$ immediately before item $\hat{\pi}[i+1]$.

Proof. By construction, $\tilde{\sigma}$ is a feasible solution. What remains to show, is that $z(\tilde{\sigma}) \leq z(\hat{\sigma})$.

Let compute the schedule of both solutions $\tilde{\sigma}$ and $\hat{\sigma}$ as described in (3)- (6). Since some items of order $o_{\hat{j}}$ are picked before item $\hat{\pi}[i+1]$ and since $s^\dagger$ also belongs to order $o_{\hat{j}}$:

$$r(s^\dagger) \leq C(\hat{\pi}[i]) \tag{15}$$

Let denote item $\hat{\pi}[i+1]$ as $s^\star$. Observe that in solution $\tilde{\sigma}$, item $s^\star$ is picked immediately after visiting location $s^\dagger$. Furthermore, in both $\tilde{\sigma}$ and $\hat{\sigma}$, items $\hat{\pi}[i], s^\star$ and $s^\dagger$ are in the same batch and the visiting sequences of the first i items are the same in $\tilde{\sigma}$ and $\hat{\sigma}$. By applying (6) and (4) in the cases of a *pushcart* and a *robotic cart*, respectively, the completion time of item $s^\star$ in solution $\tilde{\sigma}$ is:

$$\begin{aligned} C(s^\star, \tilde{\sigma}) =& \max\{C(s^\dagger) + \frac{1}{v}d(s^\dagger, s^\star), r(s^\star)\} + t^p & (16)\\ =& \max\{\max\{C(\hat{\pi}[i]) + \frac{1}{v}d(\hat{\pi}[i], s^\dagger), r(s^\dagger)\} + t^p + \frac{1}{v}d(s^\dagger, s^\star), r(s^\star)\} + t^p & (17)\\ =& \max\{C(\hat{\pi}[i]) + \frac{1}{v}d(\hat{\pi}[i], s^\dagger) + t^p + \frac{1}{v}d(s^\dagger, s^\star), r(s^\star)\} + t^p & (18)\\ \leq& r(s^\star) + t^p, & (19) \end{aligned}$$

where the equality (17) follows from (15) and the subsequent inequality (18) follows from (14) and the following observation:

$$\frac{1}{v}(d(\hat{\pi}[i], s^\dagger) + d(s^\dagger, s^\star)) \leq \frac{1}{v}2(W+L) \tag{20}$$

which holds by the bounded dimensions of the warehouse, i.e, the maximum distance between two points in the warehouse equals the warehouse length plus the warehouse width, $L + W$.

Similarly, by applying (4) and (6) in the case of a robotic cart and pushcart, respectively, the completion time of item $s^\star$ in solution $\hat{\sigma}$ cannot be smaller, since:

$$C(s^\star, \hat{\sigma}) = \max\{C(\hat{\pi}[i]) + \frac{1}{v} \cdot d(\hat{\pi}[i], s^\star), r(s^\star)\} + t^p \geq r(s^\star) + t^p \tag{21}$$

In other words:

$$C(s^\star, \tilde{\sigma}) \leq C(s^\star, \hat{\sigma}) \tag{22}$$

After applying (4) or (6) – depending on the type of cart – to compute the completion times of the remaining items and observing that the distances are metric and that the sequences of the visiting locations after picking $s^\star$ coincide in $\tilde{\sigma}$ and $\hat{\sigma}$, but $\tilde{\sigma}$ excludes the location of the already picked item $s^\dagger$, we receive that $z(\tilde{\sigma}) \leq z(\hat{\sigma})$.

□

We denote the set of the weakly dominated feasible solutions $\hat{\sigma}$ described by Lemma 1 as D_1.

Note that in the case of a *robotic cart*, the dominating feasible solution $\tilde{\sigma}$ of Lemma 1 might be dominated itself by another feasible solution in the reduced solution space, if it fulfills the characteristics of Proposition 1.

In Proposition 3.1, we have established the one-to-one correspondence between feasible solutions of an instance I and the paths from the initial state to the terminal state in the corresponding state graph. Lemma 2 uses this relation to complete the proof.

Lemma 2. *Consider a feasible transition $x_k \in X(\Theta_k)$ from a state $\Theta_k = (s_k, m^o, S^{batch}, O^{pend})$ with $|S^{batch}| \geq 1$, that dictates a next picking location $s_{k+1} \in o_j, o_j \in O^{pend}$ with the following property*

$$r_j \geq \Omega(\Theta_k) + \frac{1}{v} \cdot 2(W + L) + t^p \tag{23}$$

Then, any path from the initial to the terminal state in the state graph that involves x_k either belongs to the class of dominated solutions D_1 described by Lemma 1, or is dominated by a solution of class D_1 .

Proof. First, note that the item s_{k+1} of Lemma 2 corresponds to item $\hat{\pi}[i+1]$ in Lemma 1, and that $r(s_{k+1}) = r_j$ given that $s_{k+1} \in o_j$. Because $|S^{\text{batch}}| \geq 1$, there exists some order $o_{\hat{j}}$, some items of which are picked before item s_{k+1}, and some items of which are picked after s_{k+1}, in *any* path through the state graph that uses transition x_k.

By definition, the value $\Omega(\Theta_k)$ is the shortest time to reach state Θ_k through a path of feasible transitions from the initial state in the state graph. Let P^{shortest} be such a shortest path of transitions to state Θ_k.

First, consider any path from the initial to the terminal state of the state graph that reaches state Θ_k through P^{shortest}, followed by x_k. By definition, the completion time of s_k in the corresponding solution $\hat{\sigma}$ to such a path is $C(s_k, \hat{\sigma}) = \Omega(\Theta_k)$. Thus, because of (23), property (14) also holds for $\hat{\sigma}$ and therefore, $\hat{\sigma}$ belongs to the set of weakly dominated solutions D_1.

Second, consider any path that reaches state Θ_k through a path different from P^{shortest} from the initial state, and then uses transition x_k and continues through some path P^{end} of transitions toward the terminal state. By definition, the solution corresponding to such a path is dominated by the solution $\hat{\sigma}$ associated with path ($P^{\text{shortest}}, x_k, P^{\text{end}}$) which belongs to the set D_1. □

4.2 Proof of Proposition 4

We proceed along similar lines as in Section 4.1. For an arbitrary instance I of OBSRP-R with a *pushcart*, we use Lemma 3 to define the set D_2 of feasible solutions for I that are weakly dominated by another feasible solution. In the second step, Lemma 4 states that every path from the initial state to the terminal state in the constructed state graph, which involves a transition $x_k, k \in [n^i - 1]$ described by Proposition 4, is associated to a solution which is either in set D_2, or, which is weakly dominated by a solution in set D_2. Thus, by definition, transition x_k is weakly dominated.

Lemma 3. *Consider any feasible solution $\hat{\sigma}$ with a visiting sequence of picking locations $\hat{\pi}$ and the respective sequence of batches $\hat{\pi}^{batches}$ such that:*

- *Some item $\hat{\pi}[i+1]$ that belongs to some batch $\hat{B}_l \in \hat{\pi}^{batches}$ has a late release date:*

$$r(\hat{\pi}[i+1]) \geq C(\hat{\pi}[i]) + \frac{1}{v} \cdot 2(W + L) \tag{24}$$

- *This batch* $\hat{B}_l = \hat{B}_l^I \cup \hat{B}_l^{II}$ *can be partitioned into two sets of orders* $\hat{B}_l^I$ *and* $\hat{B}_l^{II}$ *such that*
  - $\hat{B}_l^I$ *and* $\hat{B}_l^{II}$ *are picked subsequently in* $\hat{\pi}^{\hat{B}_l} = (\hat{\pi}^{\hat{B}_l,I}, \hat{\pi}^{\hat{B}_l,II})$*,*
  - *item* $\hat{\pi}[i+1]$ *is the first item in the sequence* $\hat{\pi}^{\hat{B}_l,II}$*.*

Then the following solution $\tilde{\sigma}$ *weakly dominates solution* $\hat{\sigma}$*:*

- *Orders of* $\hat{B}_l^I$ *and* $\hat{B}_l^{II}$ *are picked in separate batches, whereas the remaining batches and their sequence remain the same:* $\pi^{batches}(\tilde{\sigma}) = (\hat{B}_1, \ldots, \hat{B}_{l-1}, \hat{B}_l^I, \hat{B}_l^{II}, \hat{B}_{l+1}, \ldots)$
- $\pi(\tilde{\sigma}) = \hat{\pi}$*, i.e. the sequences of picking locations coincide.*

Proof. The proof proceeds along the same lines as the proof of Lemma 1. By construction, $\tilde{\sigma}$ is a feasible solution. What remains to show, is that $z(\tilde{\sigma}) \leq z(\hat{\sigma})$.

Let denote item $\hat{\pi}[i+1]$ as $s^\star$. We apply equations (6) and notice that the expression $\frac{1}{v} \cdot 2(W+L)$ is an upper bound for the time needed by the picker to start in any location of the warehouse, move to the depot and, afterward, move to any other location in the warehouse. Given this, we receive that $C(s^\star, \tilde{\sigma}) \leq C(s^\star, \hat{\sigma})$. Relation $z(\tilde{\sigma}) \leq z(\hat{\sigma})$ immediately follows from (6), metric distance measure $d()$, and the definition of the objective function (2). □

Let denote the set of weakly dominated solutions $\hat{\sigma}$ introduced in Lemma 3 as D_2. Lemma 4 associates each path through the state graph that involves a transition x_k described by Proposition 4 to a weakly dominated solution and thereby closes the proof.

Lemma 4. *Consider a feasible transition* $x_k \in X(\Theta_k)$ *from a state* $\Theta_k = (s_k, m^o, \{\}, O^{pend})$ *with* $m^o \geq 1$*, that dictates a next picking location* $s_{k+1} \in o_j, o_j \in O^{pend}$ *with the following property*

$$r_j \geq \Omega(\Theta_k) + \frac{1}{v} \cdot 2(W+L) \tag{25}$$

Then, any path from the initial to the terminal state in the state graph that involves x_k *either belongs to the set of weakly dominated solutions* D_2 *described by Lemma 1, or is weakly dominated by a solution in the set* D_2 *.*

Proof. First, note that the item s_{k+1} of Lemma 4 corresponds to item $\hat{\pi}[i+1]$ in Lemma 3, and that $r(s_{k+1}) = r_j$ since $s_{k+1} \in o_j$. Given that $S^{\text{batch}} = \{\}$ and $m^o \geq 1$ in state Θ_k, item s_{k+1} belongs to batch $\hat{B}_l$ that can be decomposed as follows: $\hat{B}_l = \hat{B}_l^I \cup \hat{B}_l^{II}$, where $\hat{B}_l^I$ is a set of orders for which all items have already been picked by the time state Θ_k is reached, and $\hat{B}_l^{II}$ is a set of completely unprocessed orders at state Θ_k, one of which includes item s_{k+1}.

Using the same arguments as in the proof of Lemma 2, each path from the initial to the terminal state of the state graph that involves transition $x_k \in X(\Theta_k)$ completes the picking of item s_k either at time $C(s_k, \hat{\sigma}) = \Omega(\Theta_k)$ or at a later time. In the first case, property (25) translates to property (24), and thus the path is associated with a weakly dominated solution in the set D_2. In the second case, the solution associated with the path is weakly dominated by a solution in D_2, see Lemma 2 for an explicit construction. □

4.3 Proof of Proposition 5

Again, the proof of Proposition 5 follows the same lines as the one presented in Section 4.1. We will limit the exposition of the proof to the case of a *robotic cart*. The proof for the case of a *pushcart* follows analogously.

Consider an instance I for OBSRP-R. Lemma 5 identifies a set D_3 of feasible solutions, that are weakly dominated by another feasible solution.

Lemma 5. *Consider any feasible solution* $\hat{\sigma}$ *with a visiting sequence of picking locations* $\hat{\pi} = (\hat{\pi}^I, \hat{\pi}^{II})$ *and the respective sequence of batches* $\hat{\pi}^{batches}, |\hat{\pi}^{batches}| = \hat{f} \in \mathbb{N}$ *such that:*

- *The first subsequence* $\hat{\pi}^I$ *contains all the items of orders* $o_j \in \hat{B}_1 \cup \hat{B}_2 \cup \ldots \cup \hat{B}_l$ *and the second subsequence* $\hat{\pi}^{II}$ *contains the items of the remaining orders* $o_j \in \hat{B}_{l+1} \cup \ldots \cup \hat{B}_{\hat{f}}$ *for some* $l \in [\hat{f}-1]$*. Let denote the last location of the first subsequence* $\hat{\pi}^I$ *as* $\hat{\pi}[i]$ *and the first location of the second subsequence as* $\hat{\pi}[i+1]$*, respectively.*

- *There exists an order $o_{\hat{j}} \in \hat{B}_{l+1} \cup \ldots \cup \hat{B}_{\hat{f}}$, whose items are picked in the second subsequence $\hat{\pi}^{II}$ in $\hat{\sigma}$, which satisfies the following relation:*

$$r(\hat{\pi}[i+1]) \geq \max\{C(\hat{\pi}[i]), r_{\hat{j}}\} + \frac{1}{v} \cdot UB(o_{\hat{j}}) + |o_{\hat{j}}| \cdot t^p, \tag{26}$$

where $UB(o_{\hat{j}})$ is an upper bound for the minimum travel distance to start at the picking location $\hat{\pi}[i]$, pick all items in order $o_{\hat{j}}$ and end in any fixed but arbitrary location in the warehouse, which can be computed as described in Section 4.4 in the Appendix.

Then the following solution $\tilde{\sigma}$ is feasible and weakly dominates $\hat{\sigma}$:

- *$\pi(\tilde{\sigma}) = (\hat{\pi}^I, \pi^{\hat{j}}, \pi^{-\hat{j}})$ with $\pi^{\hat{j}}$ representing a visiting sequence of the picking locations in order $o_{\hat{j}}$ of travel distance at most $UB(o_{\hat{j}})$, and $\pi^{-\hat{j}}|\hat{\pi}^{II}$ representing the subsequence of $\hat{\pi}^{II}$ after removing all the items of order $o_{\hat{j}}$;*
  In other words, solution $\tilde{\sigma}$ picks the items of order $o_{\hat{j}}$ directly after $\hat{\pi}^I$ and then resumes visiting the remaining locations in the same order as in $\hat{\sigma}$.
- *$\pi^{batches}(\tilde{\sigma}) = (\hat{B}_1, \ldots, \hat{B}_l, \{o_{\hat{j}}\}, \hat{B}_{l+1} \setminus \{o_{\hat{j}}\}, \ldots \hat{B}_{\hat{f}} \setminus \{o_{\hat{j}}\})$,*
  i.e., in solution $\tilde{\sigma}$, order $o_{\hat{j}}$ is collected as a separate batch directly after batch $\hat{B}_l$.

Proof. By construction, $\tilde{\sigma}$ is a feasible solution, for instance, $\pi^{\text{batches}}(\tilde{\sigma})$ represents a mutually disjoint partition of the orders into batches and each batch contains at most c orders. What remains to show, is that $z(\tilde{\sigma}) \leq z(\hat{\sigma})$.

Let compute the schedule of both solutions $\tilde{\sigma}$ and $\hat{\sigma}$ as described in (3) and (4). Let denote item $\hat{\pi}[i+1]$ as $s^\star$. Observe that in solution $\tilde{\sigma}$, item $s^\star$ is picked after all the items of order $o_{\hat{j}}$ are collected in some sequence $\pi^{\hat{j}}$. By applying (4) and observing that the sequences of visiting locations for the first i items are the same in $\tilde{\sigma}$ and $\hat{\sigma}$, the completion time of this item in solution $\tilde{\sigma}$ is:

$$C(s^\star, \tilde{\sigma}) \leq \max\{\max\{C(\hat{\pi}[i]), r_{\hat{j}}\} + \frac{1}{v} d(\hat{\pi}[i], \pi^{\hat{j}}, s^\star) + |o_{\hat{j}}| \cdot t^p; r(s^\star)\} + t^p \tag{27}$$

$$\leq \max\{\max\{C(\hat{\pi}[i]), r_{\hat{j}}\} + \frac{1}{v} \cdot UB(o_{\hat{j}}) + |o_{\hat{j}}| \cdot t^p; r(s^\star)\} + t^p \tag{28}$$

$$= r(s^\star) + t^p, \tag{29}$$

where $d(\hat{\pi}[i], \pi^{\hat{j}}, s^\star)$ is the distance traveled to pick the items of $o_{\hat{j}}$ starting from $\hat{\pi}[i]$ and ending in $s^\star$ in the sequence $\pi^{\hat{j}}$.

By applying (4), the completion time of item $s^\star$ in solution $\hat{\sigma}$ cannot be smaller, since:

$$C(s^\star, \hat{\sigma}) = \max\{C(\hat{\pi}[i]) + \frac{1}{v} \cdot d(\hat{\pi}[i], s^\star), r(s^\star)\} + t^p \geq r(s^\star)\} + t^p \tag{30}$$

In other words:

$$C(s^\star, \tilde{\sigma}) \leq C(s^\star, \hat{\sigma}) \tag{31}$$

After applying (4) to compute the completion times of the remaining items and observing that the distances are metric and that the sequences of the visiting locations after picking $s^\star$ coincide in $\tilde{\sigma}$ and $\hat{\sigma}$, but $\tilde{\sigma}$ excludes the locations of the items from the already picked order $o_{\hat{j}}$, we receive that $z(\tilde{\sigma}) \leq z(\hat{\sigma})$.

□

Let denote by D_3 the set of weakly dominated solutions $\hat{\sigma}$ described by Lemma 5. The following Lemma 6 uses the one-to-one correspondence between feasible solutions for an instance and paths in the corresponding state graph from the initial- to terminal state (see Proposition 3.1), to associate any transition x_k described by Proposition 5 to a dominated solution in D_3. By definition, this closes the proof.

Lemma 6. *Consider a feasible transition $x_k \in X(\Theta_k)$ from a state $\Theta_k = (s_k, 0, \{\}, O^{pend})$, that dictates a next picking location $s_{k+1} \in o_{\bar{j}}, o_{\bar{j}} \in O^{pend}$ with the following property*

$$r_{\bar{j}} \geq \max\{\Omega(\Theta_k); r_{\hat{j}}\} + \frac{1}{v} \cdot UB(o_{\hat{j}}) + |o_{\hat{j}}| + t^p \tag{32}$$

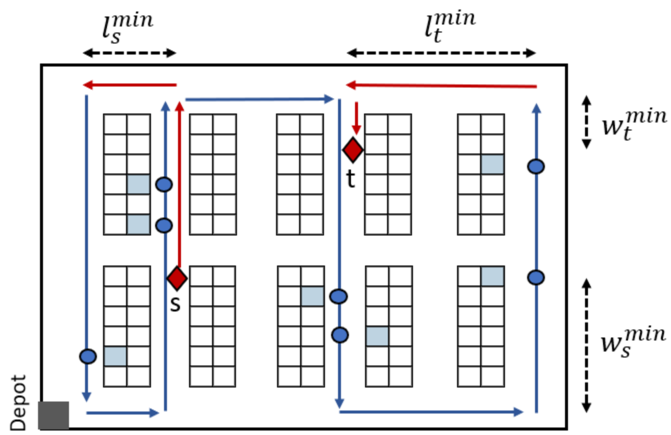


Figure 3: Illustrative example for the computation of $UB(o_j)$

Note. Picker route, starting in s and ending in t, for picking order o_j with 7 picking locations (blue circles) distributed in $n^{\text{aisles}}(o_j) = 4$ aisles, in a warehouse of length L and width W. The picker first covers distance $l_s^{min} + (W - w_s^{min})$ from s to an outer corner. Then she moves to the opposite side of the warehouse in an S-shape motion, traversing all $n^{\text{aisles}}(o_j)$ aisles that contain picking locations completely, and covering the distance of $n^{\text{aisles}}(o_j) \cdot W + L$. Finally, she moves from a corner toward t in distance $w_t^{min} + l_t^{min}$. The corners have been selected such that the total traveling distance is not greater than $UB(o_j) = (n^{\text{aisles}}(o_j) + 1)W + 2L$.

for some $o_{\hat{j}} \in O^{pend}$. Then, any path from the initial to the terminal state in the state graph that involves x_k either belongs to the set of dominated solutions D_3 described by Lemma 5, or is dominated by a solution in D_3 .

First, note that the items s_k and s_{k+1} of Lemma 6 correspond to items $\hat{\pi}[i]$ and $\hat{\pi}[i+1]$ in Lemma 1, respectively, and that $r(s_{k+1}) = r_{\bar{j}}$ given that $s_{k+1} \in o_{\bar{j}}$. Since $m^o = 0$ in $Theta_k$, a new batch is initiated by the pick of s_{k+1}.

Using the same arguments as in the proof of Lemma 2, each path from the initial to the terminal state of the state graph that involves transition $x_k \in X(\Theta_k)$ completes the picking of item s_k either at time $C(s_k, \hat{\sigma}) = \Omega(\Theta_k)$ or at a later time. In the first case, property (32) translates to property (26), and thus the path is associated to a weakly dominated solution in the class D_3. In the second case, the solution associated to the path is weakly dominated by a solution in D_3, see Lemma 2 for an explicit construction.

4.4 Upper bound for picking all items of an order

In this section, we show that the proposed upper bound $UB(o_j) = (n^{\text{aisles}}(o_j) + 1)W + 2L$ for the minimum travel distance to visit all picking locations in an order $o_j, j \in [n^o]$, starting and ending in fixed, given points in the warehouse, is indeed valid. Thereby, the expression $n^{\text{aisles}}(o_j)$ represents the number of aisles that contain the picking locations of o_j. We will show how to construct a picker route of maximum distance $UB(o_j)$ for an arbitrary order o_j, starting location s, and end ending location t.

Let us call an intersection of an outer aisle and an outer cross-aisle a *corner*. Consider the following routing strategy (see Figure 3): Move from s to some corner of the warehouse, visit all the picking locations following an S-shape route starting from this *(first)* corner and ending in the respective *last* corner, after that move to t. The S-shape route traverses all $n^{\text{aisles}}(o_j)$ aisles which contain at least one picking location completely and therefore takes time $(n^{\text{aisles}}(o_j) \cdot W + L)$. By a smart selection of the *first* corner, where we start this S-shape route, the total time ζ to reach this first corner from s and then reach t from the *last* corner, does not exceed $(W + L)$.

Let w_s^{min} and w_t^{min} be the shortest horizontal distances from s and t to reach an outer cross-aisle, respectively (see Figure 3). Observe that $w_s^{min} \leq 0.5 \cdot W$ and $w_t^{min} \leq 0.5 \cdot W$. If $w_s^{min} \leq w_t^{min}$, the picker shall move from s along w_s^{min}, then:

$$\begin{aligned} &\text{vertical component of } \zeta \\ &\leq w_s^{min} + \max\{W - w_t^{min}, w_t^{min}\} \leq W \end{aligned} \tag{33}$$

If $w_s^{min} \geq w_t^{min}$, the picker shall select the first corner such, that she can move from the last corner to t along w_t^{min}, then:

$$\begin{aligned} &\text{vertical component of } \zeta \\ &\leq \max\{W - w_s^{min}, w_s^{min}\} + w_t^{min} \leq W. \end{aligned} \tag{34}$$

By examining l_s^{min} and l_t^{min}, which are the shortest horizontal distances from s and t to reach an outer aisle, respectively, we receive that the horizontal component of $\zeta \leq L$.