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Option Prices with Stochastic Interest Rates – Black/Scholes and Ho/Lee unified

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Inhaltsverzeichnis

0. Introduction

The option pricing model by Black and Scholes (1973) and the term structure model by Ho and Lee (1986) are among the most influential models of capital market theory. While Black/Scholes consider stock option prices under the assumption of a constant deterministic interest rate, Ho and Lee were the first to model the term structure of interest rates as a stochastic object where the initial term structure concides with the empirically observed one. Whereas the original Ho/Lee–paper used a binomial setting, Heath/Jarrow/Morton (1990) could describe the limit behaviour of that model which implies normally distributed interest rates. The present paper will show that a properly enriched Black/Scholes–model and the in–the–limit Ho/Lee–model are natural companions such that an option pricing model results which is compatible with Ho/Lee term structures. The method we use is stochastic discounting. We assume the economy's asset prices to be governed by a lognormally distributed stochastic discount factor which implies a term structure compatible to the limit case of the Ho/Lee model. If we assume that the stock price at maturity is lognormally distributed we can show that the stock price follows a geometric brownian motion as it is assumed in the classical Black/Scholes–world. The combined model – consisting of the term structure and the stock price process – will be called the Black/Scholes – Ho/Lee–model. Given this model it is an easy task to compute prices for European style derivatives as, e.g., call options on such a stock. The resulting option pricing formula is a natural extension of the Black–Scholes–formula.

The paper is organized as follows: In the following section 1 we set out the basic model of the discount factor and show that it in fact implies the Ho/Lee–kind of term structures, including the forward rate process which is constantly the starting point in Heath/Jarrow/Morton type of models. Subsequently, in section 2 we construct a stock price process which is compatible both, to the Black/Scholes model and the term structure model developped in section 1. Section 3 relates the model parameters to empirically observables. Section 4 contains the theory of derivative pricing which allows to value European style derivatives on the stock in the presence of stochastic (term structures of) interest rates. In section 5 the general theory will be applied to European call options; we will present a closed form option pricing formula which is closely related to the Black/Scholes model. Section 6 is devoted to the pricing of futures contracts and presents a closed form representation for the futures price of a stock. In section 7 we will make some concluding remarks on possible generalizations and on related literature.

1. The basic model and its implications for the term structure

We consider an economy wherein the security prices are governed by a stochastic discounting factor. The basic randomness in the model consists of a probability space $(\Omega, \mathcal{A}, \mu)$ and an increasing family $\{\mathcal{A}_t \,|\, t \in \mathbb{R}_+\}$ of σ–subalgebras of $\mathcal{A}$. The stochastic discounting factor is a positive stochastic process adapted to $\{\mathcal{A}_t \,|\, t \in \mathbb{R}_+\}$. For any security which pays the $\mathcal{A}_t$–measurable random amount p_t at time t, the price at time $\tau < t$ is determined by the formula

$$p_\tau = E\left(\frac{Q_t}{Q_\tau} \cdot p_t \,\Big|\, \mathcal{A}_\tau\right) \tag{1.1}$$

provided that the security in question doesn't pay any cash in the period $]\tau, t[$ (expectation is to be taken with respect to the empirical probability measure μ).

We now specify the stochastic discounting factor as a particular function of n independent standard Wiener processes $w = (w^1, \ldots, w^n)^T$ which generate the family $\{\mathcal{A}_t \,|\, t \in \mathbb{R}_+\}$ of sub–σ–algebras. We assume

$$Q_t = e^{-t \cdot m_t - s_t \cdot \alpha^T \cdot w_t} \tag{1.2}$$

where m_t and s_t are functions of time only and $\alpha \in \mathbb{R}^n$ is a constant vector with $\|\alpha\| = \sqrt{\alpha^T \cdot \alpha} = 1$.

In order to reflect the initial term structure of interest rates $\rho_{0,t}$ we have to impose the condition

$$B_{0,t} \;:=\; e^{-t \cdot \rho_{0,t}} = E(Q_t) \tag{1.3}$$

from which we immediately get

$$Q_t = B_{0,t} \cdot e^{-\frac{1}{2}\, t \cdot s_t^2 - s_t \cdot \alpha^T \cdot w_t} \tag{1.4}$$

(for m_t this means $m_t = \rho_{0,t} + \frac{1}{2}\, s_t^2$).

For any point in time τ prior to t we can calculate

$$\frac{Q_t}{Q_\tau} = \frac{B_{0,t}}{B_{0,\tau}} \cdot e^{-\frac{1}{2}\,(t \cdot s_t^2 - \tau \cdot s_\tau^2) - s_t \cdot \alpha^T \cdot w_t + s_\tau \cdot \alpha^T \cdot w_\tau}$$

i.e.

$$\frac{Q_t}{Q_\tau} = \frac{B_{0,t}}{B_{0,\tau}} \cdot e^{-\frac{1}{2}\,(t \cdot s_t^2 - \tau \cdot s_\tau^2) + (s_\tau - s_t) \cdot \alpha^T \cdot w_\tau - s_t \cdot \alpha^T\,(w_t - w_\tau)} \tag{1.5}$$

From (1.1) and (1.5) we can conclude the implied term structure model; for a zero–bond maturing at t we get its price $B_{\tau,t}$ at τ (using (1.1))

$$B_{\tau,t} = E\left(\frac{Q_t}{Q_\tau} \,\Big|\, \mathcal{A}_\tau\right) \tag{1.6}$$

i.e., using (1.5)

$$B_{\tau,t} = \frac{B_{0,t}}{B_{0,\tau}} \cdot e^{-\frac{1}{2}\,(t\cdot s_t^2 - \tau\cdot s_\tau^2) + (s_\tau - s_t)\cdot\alpha^T\cdot w_\tau} \cdot E\left\{e^{-s_t\cdot\alpha^T\cdot(w_t - w_\tau)}\right\} \tag{1.7}$$

since $w_t - w_\tau$ and w_τ are independent by the definition of Wiener processes.

The expectation term in (1.7) amounts to

$$e^{\frac{1}{2}\,s_t^2\cdot\alpha^T\cdot\alpha\cdot(t-\tau)}$$

which yields by the norming condition $\|a\|^2 = \alpha^T \cdot \alpha = 1$

$$B_{\tau,t} = \frac{B_{0,t}}{B_{0,\tau}} \cdot e^{\frac{1}{2}\,\tau(s_\tau^2 - s_t^2)} \cdot e^{(s_\tau - s_t)\cdot\alpha^T\cdot w_\tau} \tag{1.8}$$

Inserting (1.8) into (1.5) one gets

$$\frac{Q_t}{Q_\tau} = B_{\tau,t} \cdot e^{-\frac{1}{2}\,s_t^2(t-\tau) - s_t\cdot\alpha^T\cdot(w_t - w_\tau)} \tag{1.9}$$

In terms of interest rates we get $\Big($recall that $\dfrac{B_{0,t}}{B_{0,\tau}} = e^{-(t-\tau)\,{}_0\rho_{\tau,t}}$ defines the *forward rate* ${}_0\rho_{\tau,t}\Big)$

$$\rho_{\tau,t} = {}_0\rho_{\tau,t} + \tfrac{1}{2}\,\tau\,\frac{s_t^2 - s_\tau^2}{t-\tau} \;+\; \frac{s_t - s_\tau}{t-\tau} \cdot \alpha^T \cdot w_\tau \tag{1.10}$$

which is the *term structure process* implied by the discounting factor process (1.2).

Taking the limit $t \downarrow \tau$ we obtain the process of the *instantaneous interest rate*

$$\rho_{\tau,\tau} = \rho_\tau = {}_0\rho_{\tau,\tau} + \tau \cdot s_\tau \cdot \dot{s}_\tau + \dot{s}_\tau \cdot \alpha^T \cdot w_\tau \tag{1.11}$$

A simple further calculation leads to the *forward rate process*

$${}_\tau\rho_t \;:=\; -\lim_{\Delta t \to 0} \frac{1}{\Delta\, t} \cdot \log\left(\frac{B_{\tau,t+\Delta t}}{B_{\tau,t}}\right)$$

i.e.

$${}_\tau\rho_t = {}_0\rho_t + \tau \cdot \dot{s}_t \cdot s_t + \dot{s}_t \cdot \alpha^T \cdot w_\tau \tag{1.12}$$

The forward rate process is the starting point in models which are based on the Heath/ Jarrow/Morton–approach.

The specification

$$s_\tau \;=\; \bar{s}\cdot\tau$$

yields the limit form of the Ho/Lee–model:

$$\rho_{\tau,t} = {}_0\rho_{\tau,t} + \tfrac{1}{2}\,\bar{s}^2\cdot\tau\cdot(t+\tau) + \bar{s}\cdot\alpha^T\cdot w_\tau \tag{1.13}$$

and

$$\rho_\tau = {}_0\rho_{\tau,\tau} + \tau^2\cdot\bar{s}^2 + \bar{s}\cdot\alpha^T\cdot w_\tau \tag{1.14}$$

Inserting (1.14) into (1.13) leads to

$$(\rho_{\tau,t} - {}_0\rho_{\tau,t}) = \tfrac{1}{2}\,\bar{s}^2\cdot\tau(t-\tau) + (\rho_\tau - {}_0\rho_{\tau,\tau}) \tag{1.15}$$

which is the limit form of the Ho/Lee–model (Wilhelm (1999)).

The special case of a constant function $s_t = \bar{s}$ obviously implies a situation with deterministic interest rates. In this deterministic case ($s_t = \bar{s}$) one has

$$\rho_{\tau,t} = {}_0\rho_{\tau,t} \quad \text{and} \quad \rho_\tau = {}_0\rho_{\tau,\tau} \tag{1.16}$$

for all $\tau < t$, i.e. all future spot rates equal their corresponding forward rates as seen from point in time 0.

2. The stock price model

We now introduce a stock whose terminal wealth at time t is given by

$$S_t = S_0\cdot e^{t\cdot\mu+\sigma\cdot\beta^T\cdot w_t} \tag{2.1}$$

where S_0 denotes the stock's price at point in time 0, μ and σ are some constant real numbers and β denotes a constant n–vector with $\|\beta\| = 1$. Using equation (1.1) and the discounting factor (1.9) we are now able to calculate the stock price S_τ at any point in time prior to t.

It must hold

$$S_\tau = E\left(\frac{Q_t}{Q_\tau}\cdot S_t \,\Big|\, \mathcal{A}_\tau\right) \tag{2.2}$$

and, particularly

$$S_0 = S_0 \cdot E\left(Q_t \cdot e^{t\cdot\mu+\sigma\cdot\beta^T\cdot w_t}\right) \tag{2.3}$$

i.e.

$$E\left(Q_t \cdot e^{t\cdot\mu+\sigma\cdot\beta^T\cdot w_t}\right) = 1 \tag{2.4}$$

We rewrite (2.1) a little bit and arrive at

$$S_t = S_0 \cdot e^{t\cdot\mu+\sigma\cdot\beta^T(w_t-w_\tau)} \cdot e^{\sigma\cdot\beta^T\cdot w_\tau} \tag{2.5}$$

for any $\tau < t$. Combining (2.5) and (1.9) we get

$$\frac{Q_t}{Q_\tau} \cdot S_t = S_0 \cdot B_{\tau,t} \cdot e^{\sigma\cdot\beta^T\cdot w_\tau + t\cdot\mu - \frac{1}{2}\,s_t^2(t-\tau)+(\sigma\cdot\beta^T - s_t\cdot\alpha^T)\cdot(w_t-w_\tau)} \tag{2.6}$$

Taking the conditional expectation with respect to $\mathcal{A}_\tau$ yields

$$\begin{aligned} S_\tau &= E\left(\frac{Q_t}{Q_\tau} \cdot S_t \,|\, \mathcal{A}_\tau\right) = \\ &= S_0 \cdot B_{\tau,t} \cdot e^{t\cdot\mu - \frac{1}{2}\,s_t^2(t-\tau)+\frac{1}{2}\,\|\sigma\cdot\beta - s_t\,\alpha\|^2\cdot(t-\tau)+\sigma\cdot\beta^T\cdot w_\tau} \end{aligned} \tag{2.7}$$

From $\|\sigma\cdot\beta - s_t\,\alpha\|^2 = \sigma^2 - 2\,\sigma\,s_t\cdot\beta^T\cdot\alpha + s_t^2$ we finally get

$$S_\tau = S_0 \cdot B_{\tau,t} \cdot e^{t\cdot\mu+\frac{1}{2}(\sigma^2 - 2\,\sigma\,s_t\cdot\beta^T\cdot\alpha)\cdot(t-\tau)+\sigma\cdot\beta^T\cdot w_\tau} \tag{2.8}$$

Setting $\tau = 0$ we arrive at (2.4) and find from (2.8)

$$\rho_{0,t} = \mu + \tfrac{1}{2}(\sigma^2 - 2\,\sigma\,s_t\cdot\beta^T\cdot\alpha) \tag{2.9}$$

so, ultimately, the stock price process is given by

$$S_\tau = S_0 \cdot e^{\rho_{0,t}\cdot t - \rho_{\tau,t}(t-\tau) - \frac{1}{2}\,\tau(\sigma^2-2\,\sigma\,s_t\cdot\beta^T\cdot\alpha)+\sigma\cdot\beta^T\cdot w_\tau} \tag{2.10}$$

or

$$S_\tau = S_0\,\frac{B_{\tau,t}}{B_{0,t}} \cdot e^{-\frac{1}{2}\,\tau(\sigma^2-2\,\sigma\,s_t\cdot\beta^T\cdot\alpha)+\sigma\,\beta^T\cdot w_\tau} \tag{2.11}$$

Combining (2.8) and (2.5) we can write S_t in terms of S_τ which yields

$$S_t = \frac{1}{B_{\tau,t}}\,S_\tau \cdot e^{-\frac{1}{2}\,(t-\tau)(\sigma^2-2\,\sigma\,s_t\cdot\beta^T\cdot\alpha)+\sigma\cdot\beta^T(w_t-w_\tau)} \tag{2.12}$$

This representation of the stock's terminal wealth will be used in subsequent sections.

(2.10) constitutes a consistent stock price model which is compatible with the term structure (1.10) and the stochastic discounting factor (1.2).

As a test, we specify the model for a constant function $s_t = \bar{s}$; we know from (1.10) that a non-stochastic interest rate structure with

$$\rho_{\tau,t} \;=\; {}_0\rho_{\tau,t}$$

prevails; since we have

$$\frac{B_{\tau,t}}{B_{0,t}} \;=\; \frac{1}{B_{0,\tau}}$$

from (1.8), then, the stock price model reduces to

$$S_\tau = \frac{S_0}{B_{0,\tau}} \cdot e^{-\frac{1}{2}\,\tau(\sigma^2 - 2\,\sigma\,\bar{s}\cdot\beta^T\cdot\alpha) + \sigma\cdot\beta^T\cdot w_\tau} \tag{2.13}$$

which is the basic assumption in the original Black–Scholes world when $\rho_{0,\tau}$ is assumed to be constant and β is adjusted to meet the condition

$$\rho_{0,\tau} \;=\; \|\sigma\,\beta - \bar{s}\,\alpha\|^2$$

In this Black/Scholes case there is only source of risk in the stock price.

In the general case, there are two sources of risk in the stock price: the interest rate $\rho_{\tau,t}$ which follows (1.10), and the term $\beta^T \cdot w_\tau$. The interest rate itself is a linear function of $\alpha^T \cdot w_\tau$. The two basic sources of risk $\beta^T \cdot w_\tau$ and $\alpha^T \cdot w_\tau$ are correlated by

$$\begin{aligned} & \frac{E\Big((\beta^T \cdot w_\tau)\cdot(\alpha^T \cdot w_\tau)\Big)}{\sqrt{\mathrm{var}(\beta^T \cdot w_\tau)}\cdot\sqrt{\mathrm{var}(\alpha^T \cdot w_\tau)}} \\ = \; & \frac{E\Big(\beta^T \cdot w_\tau \cdot w_\tau^T \cdot \alpha\Big)}{\tau\cdot\|\beta\|\cdot\|\alpha\|} = \beta^T \cdot \alpha \end{aligned}$$

We summarize our construction as follows: The stock price follows a lognormal process of the form

$$S_\tau = S_0 \cdot e^{\rho_{0,t}\cdot t - \rho_{\tau,t}(t-\tau) - \frac{1}{2}\,\tau(\sigma^2 - 2\,\sigma\,s_t\cdot\beta^T\cdot\alpha) + \sigma\cdot\beta^T\cdot w_\tau} \tag{2.14}$$

where the term structure of interest rates follows the following gaussian process

$$\rho_{\tau,t} = {}_0\rho_{\tau,t} + \tfrac{1}{2}\,\tau \cdot \frac{s_t^2 - s_\tau^2}{t-\tau} \; + \; \frac{s_t - s_\tau}{t-\tau}\,\alpha^T \cdot w_\tau \tag{2.15}$$

The combined model ((2.14) and (2.15)) will be called the Black/Scholes–Ho/Lee model although (2.15) is more general than the limit form of the Ho/Lee-model.

3. The process parameters

With the price process (2.14) and the term structure model (2.15) in mind it seems natural to ask how the parameters in (2.14) and (2.15) are related to empirical facts.

Lets's have a look on the (instantaneous) interest rate process (1.11), first. It is easily seen that

$$\operatorname{var}(\rho_{\tau+\Delta\tau} - \rho_\tau \,|\, \mathcal{A}_\tau) = (\dot{s}_{\tau+\Delta\tau})^2 \cdot \Delta\tau$$

holds. Hence, the function s_τ is determined by the instantaneous conditional variance of the spot rate process:

$$\lim_{\Delta\tau \to 0} \frac{\operatorname{var}(\rho_{\tau+\Delta\tau} - \rho_\tau \,|\, \mathcal{A}_\tau)}{\Delta\tau} = \dot{s}_\tau^2 \tag{3.1}$$

In an analogous manner we analyse the stock's rate of return $\log\, S_\tau$. Recalling (2.14) a simple calculation shows that

$$\begin{aligned}\operatorname{var}(\log\, S_{\tau+\Delta\tau} - \log\, S_\tau \,|\, \mathcal{A}_\tau) &= \|\sigma\,\beta - (s_t - s_{\tau+\Delta\tau})\alpha\|^2 \cdot \Delta\tau \\ &= \Big[\sigma^2 - 2\,\sigma(s_t - s_{\tau+\Delta\tau})\,\beta^T\,\alpha + (s_t - s_{\tau+\Delta\tau})^2\Big] \cdot \Delta\tau\end{aligned}$$

holds. Therefore we get (it is not hard to show that (3.2) must be valid for $\tau > t$, too):

$$\lim_{\Delta\tau \to 0} \frac{\operatorname{var}(\log\, S_{\tau+\Delta\tau} - \log\, S_\tau \,|\, \mathcal{A}_\tau)}{\Delta\tau} = \sigma^2 - 2\,\sigma(s_t - s_\tau)\beta^T\,\alpha + (s_t - s_\tau)^2 \tag{3.2}$$

as the instantaneous conditional variance of the stock return which is time–dependent in contrast to the Black–Scholes–assumptions, unless s_τ is a constant (i.e. the case of deterministic interest rates). Therefore, it doesn't make too much sense to talk about "historical volatility". In the Ho/Lee–case volatility looks like this

$$\sigma^2 + \bar{s}^2(t-\tau)^2$$

if $\beta^T\alpha = 0$ is assumed for sake of simplicity; this is quite an unsatisfactory behaviour. If the model is to be fitted to a given *time structure of volatility* of the stock return σ_τ, the volatility function s_τ has to meet

$$s_\tau = s_t - \sigma \cdot \beta^T\alpha + \sqrt{\sigma_\tau - \sigma^2(1 - (\beta^T\alpha)^2)} \tag{3.3}$$

where s_t may be chosen arbitrarily and, clearly, $\sigma_t = \sigma$ holds.

Finally, the instantaneous correlation coefficient between the stock return and the interest rate $\rho_{\tau,t}$ is given by

$$r_{\tau,t} \;=\; \operatorname{sign}(s_t - s_\tau) \cdot \frac{\sigma\,\beta^T\alpha - |s_t - s_\tau|}{\|\sigma\,\beta - (s_t - s_\tau)\,\alpha\|} \tag{3.4}$$

From (3.1), (3.2) and (3.4) it is possible – at least in principle – to estimate or specify, respectively, the interest rate related volatility function s_τ, the stock specific volatility parameter σ and the parameter $\beta^T\alpha$ which reflects the correlation between the two sources of risk which drive interest rates and stock prices. Again, the Black/Scholes–world emerges if the interest rate is deterministic (i.e. s_t is a constant).

4. The pricing of derivatives

A European style derivative is defined by a characteristic function f which relates the outcome of the derivative to the price of the underlying asset at maturity. Given such a characteristic function we can calculate the current price of the derivative by the formula:

$$c_\tau = E\left\{\frac{Q_t}{Q_\tau} \cdot f(S_t)\,|\,\mathcal{A}_\tau\right\} \tag{4.1}$$

If we denote by

$$u \;:=\; \alpha^T \cdot (w_t - w_\tau) \tag{4.2}$$

and

$$v \;:=\; \beta^T \cdot (w_t - w_\tau) \tag{4.3}$$

the random variables which determine the stochastic discount factor and the stock price as seen from point in time τ, we may rewrite (1.9) to get

$$\frac{Q_t}{Q_\tau} = B_{\tau,t} \cdot e^{-\frac{1}{2} s_t^2 (t-\tau) - s_t \cdot u} \tag{4.4}$$

and rewrite (2.12) to get

$$S_t = \frac{S_\tau}{B_{\tau,t}} \cdot e^{-\frac{1}{2}(\sigma^2 - 2\,\sigma \cdot s_t \cdot \beta^T \cdot \alpha)\cdot(t-\tau) + \sigma \cdot v} \tag{4.5}$$

Let $\varphi(u, v)$ denote the common density function of u and v then we have (u and v are jointly normally distributed, i.e. bivariate normal)

$$\varphi(u, v) = \frac{1}{2\pi(t-\tau)\sqrt{1-(\beta^T \cdot \alpha)^2}} \cdot \exp\left\{ -\frac{1}{2}\, \frac{1}{1-(\beta^T \cdot \alpha)^2} \cdot \frac{(u^2 - 2\,\beta^T\,\alpha\, u\, v + v^2)}{t-\tau} \right\} \tag{4.6}$$

The pricing equation (4.1) can now be stated as

$$c_\tau = \iint\limits_{-\infty}^{\infty} \frac{Q_t}{Q_\tau} \cdot f(S_t) \cdot \varphi(u, v) \cdot d\,u \cdot d\,v \tag{4.7}$$

where $\frac{Q_t}{Q_\tau}$ is given by (4.4) and S_t is given by (4.5); obviously c_τ is a function of S_τ and $B_{\tau,t}$ and, insofar, stochastic.

Since S_t, as seen from point in time τ, depends on v only, we may rewrite equation (4.7) and come up with

$$c_\tau = \int\limits_{v=-\infty}^{v=+\infty} f(S_t) \left(\int\limits_{u=-\infty}^{u=+\infty} \frac{Q_t}{Q_\tau} \cdot \varphi(u, v) d\,u \right) d\,v \tag{4.8}$$

In order to evaluate (4.8) we focus on the expression

$$A(v) = \frac{1}{B_{\tau,t}} \int\limits_{-\infty}^{\infty} \frac{Q_t}{Q_\tau} \varphi(u, v) d\,u \tag{4.9}$$

first.

By a boring but rather simple calculation one obtains:

$$A(v) = \frac{1}{\sqrt{2\,\pi}\,\sqrt{t-\tau}} \cdot e^{-\frac{1}{2}\;\frac{1}{t-\tau}\cdot(v+\beta^T\cdot\alpha\cdot s_t\cdot(t-\tau))^2} \tag{4.10}$$

Using the standardized normal density

$$n(x) = \frac{1}{\sqrt{2\,\pi}} \cdot e^{-\frac{1}{2}\,x^2}$$

we get

$$A(v) = \frac{1}{\sqrt{t-\tau}} \cdot n\left(\frac{v+\beta^T\cdot\alpha\cdot s_t(t-\tau)}{\sqrt{t-\tau}}\right) \tag{4.11}$$

which we call the **valuation density**. So, we get a general pricing formula for European style derivatives which reads as:

$$c_\tau = B_{\tau,t} \cdot \int\limits_{-\infty}^{\infty} f\left(\frac{S_\tau}{B_{\tau,t}} \cdot e^{-\frac{1}{2}(\sigma^2-2\,\sigma\cdot s_t\cdot\beta^T\cdot\alpha)(t-\tau)+\sigma\cdot v}\right) \cdot A(v)d\,v \tag{4.12}$$

Substituting $y = \dfrac{v}{\sqrt{t-\tau}}$ and $z = y + \beta^T \cdot \alpha \cdot s_t\sqrt{t-\tau}$ yields

$$\begin{aligned}
c_\tau &= B_{\tau,t} \cdot \int\limits_{-\infty}^{\infty} f\left[\frac{S_\tau}{B_{\tau,t}} \cdot e^{-\frac{1}{2}(\sigma^2-2\,\sigma\cdot s_t\cdot\beta^T\cdot\alpha)(t-\tau)+\sigma\sqrt{t-\tau}\cdot y}\right] \cdot n\left(y+\beta^T\cdot\alpha\cdot s_t\sqrt{t-\tau}\right)d\,y \\
&= B_{\tau,t} \int\limits_{-\infty}^{\infty} f\left[\frac{S_\tau}{B_{\tau,t}} \cdot e^{-\frac{1}{2}\,\sigma^2(t-\tau)+\sigma\cdot z\cdot\sqrt{t-\tau}}\right] \cdot n(z) \cdot dz \\
&= B_{\tau,t} \int\limits_{-\infty}^{\infty} f\left[\frac{S_\tau}{B_{\tau,t}} \cdot e^{-\frac{1}{2}\,(z-\sigma\sqrt{t-\tau})^2+\frac{1}{2}\,z^2}\right] \cdot n(z) \cdot dz
\end{aligned} \tag{4.13}$$

which is ready to be applied to special cases.

5. European call options

As the standard example we consider an European call option on the stock S which matures at time t at a striking price X. So the characteristic function reads as follows:

$$f(S) = \max\{S-X,0\} \tag{5.1}$$

It is convenient to define

$$d^* = \frac{\log\left(\frac{X \cdot B_{\tau,t}}{S_\tau}\right)}{\sigma\sqrt{t-\tau}} \;+\; \frac{1}{2}\sigma\sqrt{t-\tau} \tag{5.2}$$

so that (4.13) can be rewritten in the following way:

$$\begin{aligned} c_\tau &= S_\tau \int_{d^*}^{\infty} e^{-\frac{1}{2}\sigma^2(t-\tau)+\sigma\cdot z\sqrt{t-\tau}} \cdot n(z)dz - B_{\tau,t}\cdot X \cdot \int_{d^*}^{\infty} n(z)dz \\ &= S_\tau \int_{d^*-\sigma\sqrt{t-\tau}}^{\infty} n(z)dz - B_{\tau,t}\cdot X \cdot \int_{d^*}^{\infty} n(z)dz \end{aligned} \tag{5.3}$$

So we finally find the following option pricing formula:

$$c_\tau = S_\tau \cdot \left(1 - N\left(d^* - \sqrt{t-\tau}\,\sigma\right)\right) - B_{\tau,t}\cdot X\cdot(1-N(d^*)) \tag{5.4}$$

This formula coincides with the famous Black–Scholes–equation in spite of stochastic (term structures of) interest rates.

6. Futures prices

The stochastic discounting factor (1.9) allows to derive what we have called the "futures evaluator" elsewhere (see Wilhelm (1999)). Given a spot price process p_t the futures price will be denoted by $F_{\tau,t}$ which means the futures price of a contract written at time τ to be delivered a time t. From Cox/Ingersoll/Ross (1981) we know that the following relation holds:

$$F_{\tau,t} = E\left(\frac{Q_t}{Q_\tau}\cdot e^{\int_\tau^t \rho_\theta\cdot d\theta}\cdot p_t|\mathcal{A}_\tau\right) \tag{6.1}$$

or as a limit

$$F_{\tau,t} = \lim_{h\to 0} E\left(\frac{Q_t}{Q_\tau}\cdot e^{\sum_{i=0}^{k-1}\rho_{\tau+i\cdot h}\cdot h}\cdot p_t\Big|\,\mathcal{A}_\tau\right) \tag{6.2}$$

where $k \cdot h = t - \tau$ holds. Since we have

$$e^{\sum_{i=0}^{k-1} \rho_{\tau+i\cdot h} \cdot h} = \prod_{i=0}^{k-1} B^{-1}_{\tau+i\cdot h, \tau+(i+1)h}$$

and

$$\frac{Q_t}{Q_\tau} = \lim_{h \to 0} \prod_{i=0}^{k-1} \frac{Q_{\tau+(i+1)\cdot h}}{Q_{\tau+i\cdot h}}$$

we may rewrite the coefficient of p_t in (6.2) in the following way

$$\prod_{i=0}^{k-1} \left[\frac{Q_{\tau+(i+1)\cdot h}}{Q_{\tau+i\cdot h}} \cdot \frac{1}{B_{\tau+i\cdot h, \tau+(i+1)\cdot h}} \right]$$

By using (1.9) with $\tau \mapsto \tau + i \cdot h$ and $t \mapsto \tau + (i+1)h$ we get

$$\prod_{i=0}^{k-1} \left[e^{-\frac{1}{2} s^2_{\tau+(i+1)h} \cdot h - s_{\tau+(i+1)h} \cdot \alpha^T (w_{\tau+(i+1)h} - w_{\tau+i\cdot h})} \right]$$

$$= e^{-\frac{1}{2} \sum_{i=0}^{k-1} s^2_{\tau+(i+1)h} \cdot h - \sum_{i=0}^{k-1} s_{\tau+(i+1)h} \cdot \alpha^T (w_{\tau+(i+1)h} - w_{\tau+i\cdot h})} \tag{6.3}$$

For $\tau = 0$ we get in the limit

$$V_t \ := \ e^{\frac{1}{2} \int_0^t s^2_\theta \cdot d\theta - \int_0^t s_\theta \cdot \alpha^T \cdot d\, w_\theta} \tag{6.4}$$

which we call the futures evaluator since

$$F_{0,t} \ = \ E(V_t \cdot p_t) \tag{6.5}$$

holds.

In the more general case (6.1) we get by a simple consideration

$$F_{\tau,t} = E \left\{ \frac{V_t}{V_\tau} \cdot p_t \,\middle|\, \mathcal{A}_\tau \right\} \tag{6.6}$$

It is now a rather easy task to calculate the futures price of the stock whose terminal wealth at the delivery date t is given by (2.12) which we write in an appropriately approximate form:

$$S_t = \frac{S_\tau}{B_{\tau,t}} \cdot e^{-\frac{1}{2}(t-\tau)(\sigma^2 - 2\sigma\, s_t \cdot \beta^T \cdot \alpha) + \sigma \cdot \sum\limits_{i=0}^{k-1} \beta^T (w_{\tau+(i+1)h} - w_{\tau+i\cdot h})} \tag{6.7}$$

Applying (6.1) by using (6.3) we get

$$\begin{aligned} F_{\tau,t} \quad &= \quad \frac{S_\tau}{B_{\tau,t}} \cdot e^{-\frac{1}{2}\left[(t-\tau)(\sigma^2 - 2\sigma\, s_t \cdot \beta^T \cdot \alpha) + \int\limits_\tau^t s_\theta^2\, d\theta\right]} \\ &\qquad \cdot E\left(e^{\sum\limits_{i=0}^{k-1} (\sigma\, \beta^T - s_{\tau+(i+1)h} \cdot \alpha^T)(w_{\tau+(i+1)h} - w_{\tau+i\cdot h})} \Big| \mathcal{A}_\tau \right) \end{aligned} \tag{6.8}$$

The expectation term becomes

$$e^{\frac{1}{2} \sum\limits_{i=0}^{k-1} \|\sigma\, \beta - s_{\tau+(i+1)h} \cdot \alpha\|^2 \cdot h}$$

which tends to

$$e^{\frac{1}{2} \int\limits_\tau^t \|\sigma\, \beta - s_\theta \alpha\|^2\, d\theta} = e^{\frac{1}{2}\sigma^2 (t-\tau) - \sigma\, \beta^T\, \alpha \int\limits_\tau^t s_\theta\, d\theta + \frac{1}{2} \int\limits_\tau^t s_\theta^2\, d\theta}$$

as h tends to zero.

So we have

$$F_{\tau,t} = \frac{S_\tau}{B_{\tau,t}} \cdot e^{\sigma \cdot \beta^T\, \alpha (s_t (t-\tau) - \int\limits_\tau^t s_\theta\, d\theta)} \tag{6.9}$$

as the futures price of our stock. The exponential term in (6.9) makes the difference to the forward price $\frac{S_\tau}{B_{\tau,t}}$. Both prices coincide, on the one hand, in the case of s_t being a constant which implies deterministic interest rates; this is a well–known condition. On the other hand, the two prices also coincide in the case of a zero–correlation between the two sources of risk. (6.9) may serve as a starting point for the valuation of derivatives on the futures price of a stock.

7. Concluding remarks

The present paper has developed a model that incorporates the basic features of the option pricing results of Black/Scholes and the theory of stochastic term structures advanced by Ho/Lee. The method employed is stochastic discounting. We start from a certain discounting factor that governs all asset prices in the economy and can specify all ingredients one needs to characterize stock price and interest rate processes. In addition, the advantage of the stochastic discounting approach is that the empirical probabilities are directly used without shifting to an equivalent martingale measure. The discount factor we use seems to be the most simple one which is able to produce such a rather rich theory. On the other hand one might ask for generalization. A discounting factor of the form

$$Q_t = B_{0,t} \cdot e^{-\frac{1}{2}\int\limits_0^t \|s(t,\theta)\|^2 \, d\theta - \int\limits_0^t s(t,\theta)^T \, dw_\theta} \tag{7.1}$$

with a n–vector function $s(t,\theta)$ would be even more flexible while being more difficult to use and to specify parameters: The reulting term structure process is given by

$$\begin{aligned} \rho_{\tau,t} = \ {}_0\rho_{\tau,t} \quad &+ \quad \tfrac{1}{2}\,\frac{1}{t-\tau}\int\limits_0^\tau \|s(t,\theta) - s(\tau,\theta)\|^2 \cdot d\theta \\ &+ \quad \frac{1}{t-\tau}\int\limits_0^\tau \Big(s(t,\theta) - s(\tau,\theta)\Big)^T \cdot dw_\theta \end{aligned} \tag{7.2}$$

which is an obvious generalization of (2.15). The instantaneous spot rate looks like this

$$\rho_\tau = \ {}_0\rho_\tau + \tfrac{1}{2}\int\limits_0^\tau \tfrac{\partial}{\partial\tau}\|s(\tau,\theta)\|^2 \cdot d\theta + \int\limits_0^\tau \tfrac{\partial}{\partial\tau} s(\tau,\theta)^T \cdot dw_\theta \tag{7.3}$$

and the stock price process becomes:

$$S_\tau = S_0 \cdot \frac{B_{\tau,t}}{B_{0,t}} \cdot e^{-\frac{1}{2}\tau\left[\|\sigma\|^2 - 2\frac{1}{\tau}\sigma^T\cdot\int\limits_0^\tau s(t,\theta)d\theta\right] + \sigma^T\cdot w_\tau} \tag{7.4}$$

if

$$S_t \;=\; S_0 \cdot e^{\mu\cdot t + \sigma^T\cdot w_t} \tag{7.5}$$

is assumed. The process (7.2) adds some additional structure since it allows differentiated correlations among interest rates of different maturities:

$$\mathrm{corr}(\rho_{\tau,t}\,,\ \rho_{\tau,t^*}) =$$

$$= \frac{\int_0^\tau \left[\left(s(t,\theta) - s(\tau,\theta) \right)^T \cdot \left(s(t^*,\theta) - s(\tau,\theta) \right) \right] d\theta}{\sqrt{\int_0^\tau \|s(t,\theta) - s(\tau,\theta)\|^2 \, d\theta} \cdot \sqrt{\int_0^\tau \|s(t^*,\theta) - s(\tau,\theta)\|^2 \, d\theta}} \tag{7.6}$$

Furthermore, it is not hard to calculate a valuation density in the spirit of (4.10) in this case, too. However, to keep things as simple as possible we do not follow this line further.

The present paper is related to the work of Milterson/Schwartz (1998) who derive, in their gaussian case, results very similar to ours using the equivalent martingale approach and the Heath/Jarrow/Morton methodology for modeling interest rates. The stochastic discounting approach used in this paper has the advantage of keeping mathematics very simple in the gaussian case and making direct use of empirical probabilities throughout the computations.

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