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Some Economic Remarks on Arbitrage Theory

Bernhard Nietert and Jochen Wilhelm* **

Abstract:

Today's primarily mathematically oriented arbitrage theory does not address some economically important aspects of pricing. These are, first, the implicit conjecture that there is "the" price of a portfolio, second, the exact formulation of no–arbitrage, price reproduction, and positivity of the pricing rule under short selling constraints, third, the explicit assumption of a nonnegative riskless interest rate, and fourth, the connection between arbitrage theory (that is almost universal pricing theory) and special pricing theories. Our article proposes the following answers to the above issues: The first problem can be solved by introducing the notion of "physical" no–arbitrage, the second one by formulating the concept of "actively" traded portfolios (that is non–dominated portfolios) and by requiring that there is a minimum price for actively traded portfolios and therefore for every admissible portfolio, and the third one by combining the "invisible" asset "cash" with the idea of actively traded portfolios – a riskless asset with a rate of return less than zero can never be actively traded in the presence of cash. Finally, the connection between arbitrage theory and special pricing theories ("law–of–one–price–oriented" and "utility–oriented" pricing) consists in the fact that special pricing theories merely concretize arbitrage theory using different assumptions.

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Some Economic Remarks on Arbitrage Theory

1 Preliminaries

1.1 Introduction to the problem

Financial assets, especially derivatives (options, swaps, futures et cetera), are used in numerous economic transactions. To successfully manage these transactions, we need to know the relationship between prices of these assets and their influencing factors. On a theoretical basis, pricing theory claims to deliver exactly these "fair" prices.

However, today's pricing literature is pretty heterogeneous and has a primarily mathematical focus. The reason for this situation is that, starting from the crucial paper *Ross* (1978), the main research effort of arbitrage theory has been laid on the mathematically exact formulation of no–arbitrage conditions: *Harrison/Kreps* (1979) and *Harrison/Pliska* (1981) discover the relationship between no–arbitrage and martingale measures thereby working out the qualifications no–arbitrage imposes on trading strategies in continuous time, *Kreps* (1981) adapts the notion of no–arbitrage to an infinite number of assets, and *Delbaen* (1992) as well as *Delbaen/Schachermayer* (1994) apply Kreps's results to derive sophisticated existence conditions for martingale measures[1].

Arguing in a mathematically exact way and specifying no–arbitrage conditions with the help of (equivalent) martingale measures undoubtedly is deserving because it has made economists aware of possible technical problems and has made accessible to them the numerous tools of mathematical stochastics. But, it also entails substantial costs: It does not discuss implied economic assumptions and obscures basic economic relations, which, in our mind, leads to the following four problems:

1. Existence of "the" price of an asset:

   Have a look at the following market:

		states of the world	
asset	price	state 1	state 2
A_0	100	100	100
A_i	75	110	90
A_j	75	80	120

Table 1: payoffs and prices on market I

[1]A more extensive overview over the development of the mathematical literature can be found in Musiela/Rutkowski (1997), p. 246-247.

We are given three assets, two states of the world and one price for each asset. To be more precise, we have one price for each asset irrespective of the transactions' circumstances, for example, whether the asset is listed on several exchanges, or whether we buy high or low quantities and so on.

However, the assumption of "the" price may incorrectly specify investors' opportunity sets in that it contains the (implicit) no–arbitrage condition that those circumstances are irrelevant. The fundamental texts of arbitrage theory (*Ross* (1978), *Harrison/Kreps* (1979), and *Harrison/Pliska* (1981)) only embody hints at scenarios where transactions' circumstances do not matter, but no systematic analysis: First, when *Ross* (1978)[2] states, that the multiple of a payoff must coincide with the multiple of its price. Second, when *Harrison/Pliska* (1981)[3] define a linear price functional thus assuming that the number of units bought and sold have no price influence. However, both statements are not sufficiently precise because they merely cover a small amount of possible circumstances.

Therefore, we have to ask: "What are the exact conditions under which "the" price of an asset exists irrespective of the transactions' circumstances?

2. Formulation of no–arbitrage, price reproduction, and positivity of the pricing rule under short selling constraints:

Accept for a while that there is "the" price of an asset. Nevertheless there is another problem with market I, which is reflected in the pricing rule as given by state prices ϕ of this market segment. To see this, recall that, for a linear pricing rules, the price of each asset k should equal its discounted payoff, which reads when applied to market I: $P_k = Z_k(S_1)\phi(S_1) + Z_k(S_2)\phi(S_2)$ (with $k = 0, i, j$). Since we have three assets, we get an equation system with three equations and two unknown state prices resulting in the following potential solutions:

Combination of	yields
A_0 and A_i	$\phi_{S_1} = -\frac{3}{4}$ and $\phi_{S_2} = \frac{7}{4}$
A_0 and A_j	$\phi_{S_1} = -\frac{1}{8}$ and $\phi_{S_2} = \frac{9}{8}$
A_i and A_j	$\phi_{S_1} = \frac{3}{8}$ and $\phi_{S_2} = \frac{3}{8}$

Table 2: state prices on market I

As can be seen easily, state prices neither reproduce the prices of all assets nor are they positive, indicating that the market is not free of arbitrage according to the first

[2]See Ross (1978), p. 457.
[3]See Harrison/Pliska (1981), p. 226.

fundamental theorem of asset pricing[4]. This is obvious because there is a difference arbitrage in the market: Choose $N_i = -\frac{2}{3}N_0$ and $N_j = -\frac{1}{3}N_0$ to obtain $-25 \cdot N_0 > 0$ at $t = 0$ without having to pay anything at $t = 1$ (the payoffs in state 1 as well as state 2 equal zero).

Now assume that A_0 is subject to short selling constraints. This institutional restriction prevents us from effectively carrying out the above arbitrage and the market will be in fact free of arbitrage. For this reason, no–arbitrage under short selling constraints might be thought of as absence of a difference arbitrage combined with an additional test of binding short selling constraints. According to *Wilhelm* (1987) and *Jouini/Kallal* (1995a) we know that this adapted no–arbitrage criterion yields a positive and sublinear[5] price functional.

The literature so far can only rule out non–positive price functionals. It does not, however, provide an answer to the question, whether the positive state prices derived from A_i and A_j constitute a price functional valid for the market as a whole and how to deal with A_0, an asset definitely not priced by this price functional.

Therefore, we have to ask: "How can we find a positive price functional on markets subject to short selling constraints that nevertheless contains price information for all assets in the market?"

3. Endogenous justification of a nonnegative riskless rate:

Imagine a market II quite similar to market I:

		states of the world	
asset	price	state 1	state 2
A_a	$111\frac{1}{9}$	100	100
A_i	$108\frac{8}{9}$	110	90
A_j	$115\frac{5}{9}$	80	120

Table 3: payoffs and prices on market II

Contrary to market I, market II has unique and positive state prices $\phi_{S_1} = \frac{4}{9}$ and $\phi_{S_2} = \frac{2}{3}$. According to the first fundamental theorem of asset pricing it is free of

[4]According to Dybvig/Ross (1992), p. 44, the phrase "first fundamental theorem of asset pricing" describes the fact that no–arbitrage and the existence of a positive linear price functional are equivalent statements.

[5]"Sublinear" means subadditive, that is $\phi(Z^* + \hat{Z}) \leq \phi(Z^*) + \phi(\hat{Z})$, and positive homogeneous of degree one, that is $\phi(\alpha Z^*) = \alpha\phi(Z^*)$ for $\alpha > 0$. Moreover, a functional ϕ is called "positive" if $\phi(Z) \geq 0$ holds as long as $Z \geq 0$ is true.

arbitrage and the problem of market I seems to be gone. – Yet, a look at the riskless rate implied by this market reveals that it is negative: $r = \dfrac{1}{\phi_{S_1} + \phi_{S_2}} - 1 = -10\%$.

The existing arbitrage theory and especially the first fundamental theorem of asset pricing obviously is unable to guarantee a nonnegative riskless rate because it derives merely statements about the position of the riskless rate (or more general: of the numéraire) relative to other asset prices, but not with respect to an absolute lower bound. The literature acknowledges that such an outcome is economically not reasonable and mostly[6] assumes a positive riskless rate explicitly.

Therefore, something seems to be missing in market II and we have to ask: "How can we endogenously assure that the riskless rate is nonnegative?"

4. Relationship between arbitrage theory and special pricing theories:

   Mathematical finance regards arbitrage theory as an autonomous area of research. As applied to our examples this means, mathematical finance claims and proves the existence of a price functional, but does not discuss the economic forces behind it; this task is fulfilled by special pricing theories. The only exception are *Musiela/Rutkowski* (1997)[7], who derive the Black/Scholes option pricing formula from martingale considerations, thereby connecting martingales, that is state price densities, with the preference free hedging–methodology of Black/Scholes. – On the other hand, economists have done a more elaborate job in relating their special pricing results to arbitrage theory. Thanks to *Schöbel* (1995) we know that the numerous variants of special pricing theories are all derived from two basic forms: law–of–one–price– and utility–oriented pricing. *Cox/Ross/Rubinstein* (1979)[8] calculated state prices in the binomial option pricing model with the help of the duplication portfolio, thus examining one special case of law–of–one–price–based valuation. *Duffie* (1996)[9] proved that utility-based pricing specifies indeed the state prices of arbitrage theory.

   However, the analysis of general connections between law–of–one–price–based pricing and arbitrage theory as well as between law–of–one–price– and utility–oriented pricing is still missing.

   Therefore, we have to ask: "Is there a generally valid relation between arbitrage theory and the two basic versions of special pricing theories?"

1.2 Introduction to the solution technique

The above problems demonstrate that the existing arbitrage theory is not as economically founded as it should be. For that reason, in our paper we strive to improve both the economic foundation and interpretation of arbitrage theory by solving these four problems.

[6]See for example Harrison/Pliska (1981), p. 216.
[7]See Musiela/Rutkowski (1997), p. 113.
[8]See Cox/Ross/Rubinstein (1979), p. 240.
[9]See Duffie (1996), pp. 7-8.

To achieve this goal we take, methodologically, a point of view complementary to that of mathematical finance: The latter uses a crude economic framework to obtain mathematically sophisticated no–arbitrage conditions. We, on the opposite, argue within a mathematically crude one–period model, yet we do not impose further restrictions on the economic environment.

Based thereon, we provide the following answers to the four questions raised above:

1. As a solution to the assumption of "the" price of an asset we will introduce the concept of **physical no–arbitrage**:

   Physical no–arbitrage focuses on the price of each assets and, thus, relates to physical securities. It gives an exact description of the facts and the implications making the circumstances of a transaction (numbers purchased or sold, numbers of exchanges the asset is listed at et cetera) irrelevant. It does not assume that there is, for instance, just one exchange thereby ignoring buying and selling assets at different exchanges and hence an integral part of investors' trading possibilities. Physical no–arbitrage consequently does not impose artificial restrictions on investors' opportunity sets.

2. As a solution to the formulation of no–arbitrage, price reproduction, and positivity of the price functional under short selling constraints we will impose a more precise notion of **economical no–arbitrage**:

   As opposed to physical no–arbitrage, economical no–arbitrage connects assets' payoffs to their prices (payoff orientation). Economical no–arbitrage under short selling constraints means that it should not be feasible to acquire **actively traded** portfolios and hence any admissible portfolios at an arbitrary low price. This form of a no–arbitrage condition relies first on the notion of **arbitration of exchange**. Arbitration of exchange[10] rests on the economic principle and describes nothing else than the desire of getting a given payoff at the minimum price possible. Second, it uses the concept of, what we will call, **actively traded** portfolios. The term "actively traded" builds on dominance relations between assets: Purchasing an asset at a higher price, whose payoff is dominated by another asset's payoff, cannot be a reasonable choice. Thus, there is no demand for the dominated asset, it is not actively traded. Economical no–arbitrage now means that the price of a dominated payoff must always be below the price of the dominating payoff (actively traded portfolio), which yields a lower price bound for any admissible portfolio.

   From the above characterization of no–arbitrage it follows the existence of a price functional that determines for all (attainable) payoffs a lower price bound. This price functional reproduces the price for actively portfolios, for portfolios not actively traded it defines a lower price bound. However, under short selling constraints this price functional is not necessarily linear, which means it cannot be interpreted

[10] To clarify the phrase "arbitration of exchange", we go back to the institutionally based literature. According to Munn (1983), p. 54 "arbitration of exchange" means "a calculation based on rates of exchange to determine the difference in value of a given currency in three different places or markets, particularly when made with a view of determining the cheapest way of making a remittance between two countries".

as given by state prices. A linear price functional and thus state prices only apply to portfolios not subject to short selling constraints irrespective of whether these constraints are binding or not.

3. As a solution to the endogenous justification of a nonnegative riskless rate we will "reactivate" the "invisible", but nevertheless permanently present, asset **cash**:

   Cash is an "asset" that is always available. Every amount of money that is neither consumed nor invested (in the strict sense) today is automatically transferred to tomorrow in cash. Due to this automatism cash seems to be invisible, but nevertheless belongs always to investors' investment opportunity set.

   Using this correct description of investors' investment opportunity set, we infer that a positive riskless rate dominates cash and, hence, cash will not be actively traded. A negative riskless rate, however, is dominated by the zero "interest" rate of cash. As a consequence, a riskless asset with a negative riskless rate is not actively traded. From both arguments it follows that the riskless rate has to be nonnegative.

4. As an answer to the question of how arbitrage theory and special pricing theories are related, we will demonstrates that under the absence of short selling constraints special pricing theories specify the state prices (price functional) of arbitrage theory:

   By constituting the structure of fair asset prices, arbitrage theory undoubtedly is a pricing theory albeit a quite universal one. Arbitrage theory, however, does not compute the price functional by specifying, for example, state prices; this task is carried out by special pricing theories like "law–of–one–price–oriented" and "utility–oriented" pricing. Yet, law–of–one–price–oriented and utility based pricing do not conflict with each other, but simply rely on different sets of assumptions: Law–of–one–price–oriented pricing rests on the spanning assumption, whereas utility–oriented pricing is based on the expected utility principle. Moreover, whenever spanning of a payoff is possible, law–of–one–price– and utility–oriented pricing must and will result in the same pricing rule.

The objective of our paper – to lay out the economical foundation and interpretation of arbitrage theory – calls for the following structure of our considerations: In chapter 2 we will develop the framework of our model in a rather non-technical way and define no–arbitrage (under short selling constraints). Chapter 3 will develop the solutions to our four problems posed above. The paper will end with some concluding remarks (chapter 4) and a technical appendix.

2 No-arbitrage under short selling constraints

From our introduction it has become clear that the existing arbitrage theory cannot satisfactorily answer at least four economically important questions. Therefore, it is quite natural to analyze arbitrage theory and its core, the no–arbitrage condition, a bit more thoroughly.

2.1 Model's framework

To this end, we go into details of our notion of a market segment; we will concentrate – as already mentioned in the introduction – on one–period financial markets with additional characteristics:

- There are (including cash) $n+1$ physically different assets, which are arbitrarily divisible and fungible. Their rights are proportional to the numbers of units held; we are only interested in the rights to obtain the payoff Z_i per unit of asset i (Z_i is a random variable). Cash is the asset indexed by zero.
- There may be different "circumstances" $u \in \{u_0, u_1, \dots\}$ under which tradable objects $\mathbf{N}$ are in fact traded at transaction price $P_u(\mathbf{N})$. "Circumstances" mean, for example, whether the asset is listed on several exchanges, or whether one buys high or low quantities, or whether one has different trading partners et cetera.
- Trade objects are portfolios (mathematically represented as $n+1$–dimensional column vectors) $\mathbf{N}$ with $\mathbf{N}^T = (N_0 \ \dots \ N_n)$ and N_i denoting the number of asset i's units involved in the transaction (the superscript ^{T} indicates transposition of vectors and matrices). For instance, trading just one unit of asset number one signifies $\mathbf{N}^T = (0 \ 1 \ 0 \ \dots \ 0)$, making individual assets "special" portfolios.
- For every (in an institutional sense) admissible portfolio $\mathbf{N}$ the portfolio $z \cdot \mathbf{N}$ will be admissible whenever $z \in \mathbb{R}^+$ holds. Moreover, short selling of a portfolio $\mathbf{N}$ will be admissible whenever $-\mathbf{N}$ is admissible. For every pair of admissible portfolios $\mathbf{N}_i$ and $\mathbf{N}_j, \mathbf{N} = \mathbf{N}_i + \mathbf{N}_j$ is admissible[11].

2.2 Definition of no–arbitrage under short selling constraints

Based on the above characterization of a financial market, we define no–arbitrage (under short selling constraints) as follows:

A market segment will be called **free of arbitrage** if there are neither **physical** nor **economical** arbitrage opportunities.

To put this definition to work, we have to examine and explain its components.

[11] Technically speaking, the set of admissible portfolios forms a convex cone with vertex 0.

2.2.1 Physical no–arbitrage

Physical no–arbitrage focuses on **the** price of each asset, and is thus related to physical securities (stock of company A, bond of company B et cetera), which means:

> Under no circumstances it is worth thinking of bundling or unbundling positions, that is for all all possible circumstances $u \in \{u_0, u_1, \dots\}$ we have
>
> $$P_u(0) = 0 \tag{2.1}$$
>
> and
>
> $$P_{u_0}(\mathbf{N}_{u_0}) = P_{u_1}(\mathbf{N}_{u_1}) + P_{u_2}(\mathbf{N}_{u_2}) + \dots$$
>
> whenever
>
> $$\mathbf{N}_{u_0} = \mathbf{N}_{u_1} + \mathbf{N}_{u_2} + \dots$$
>
> is **true**.

To give an example: Consider buying the portfolio $\mathbf{N}_{u_0}^T = (20 \quad 20 \quad 20)$ all together at one exchange (circumstance u_0) or using two transactions at different exchanges $\mathbf{N}_{u_1}^T = (15 \quad 5 \quad 15)$ and $\mathbf{N}_{u_2}^T = (5 \quad 15 \quad 5)$. Owing to physical no–arbitrage, we have to pay the same price for the direct purchase as in sum for the split purchases, that is

$$P_{u_0}\begin{pmatrix}20\\20\\20\end{pmatrix} = P_{u_1}\begin{pmatrix}15\\5\\15\end{pmatrix} + P_{u_2}\begin{pmatrix}5\\15\\5\end{pmatrix} \text{ must hold.}$$

Physical no–arbitrage obviously allows us to skip the circumstances u, under which transactions take place, and write $P(\mathbf{N})$ as **the** price of the portfolio $\mathbf{N}$.

Moreover, from our definition of physical no–arbitrage we obtain the following implications[12]:

I 1: Every asset i only has one price P_i per unit. This signifies, there is no advantage from trades using prices of the same asset at different stock exchanges.

I 2: The price P_{Pf} of the portfolio $\mathbf{N}$ equals the sum of the prices of the single assets weighted with their numbers $\left(\sum_{i=1}^{n} N_i P_i = P_{Pf}\right)$, that is there is no "volume charge or discount".

I 3: From the price P_{mult} of the multiple N_i of asset i we can unequivocally derive the price of one unit of asset i: $P_i = \dfrac{P_{\text{mult}}}{N_i}$.

[12] The proof that (I 1) to (I 3) are in fact a consequence of physical no–arbitrage can be found in the appendix.

To better assess the notion of physical no–arbitrage and its three consequences (I 1) to (I 3), look at the existing literature: *Black/Scholes* (1973)[13] briefly touch I 1 when they assume an unique option price as solution to their fundamental pricing equation. *Wilhelm* (1981)[14] formulates I 1, however without recognizing the underlying general case of physical no–arbitrage. I 2 is only partly reflected, when *Ross* (1978)[15] states that the multiple of a payoff must coincide with the multiple of its price, or *Harrison/Pliska* (1981)[16] define a linear price functional on asset returns. Yet, *Harrison/Pliska* (1981) put the cart before the horse because they present the consequence of no–arbitrage (existence of a linear connection between payoff and price) before its definition. Again, *Wilhelm* (1981) contains I 2, but once more he does not recur to physical no–arbitrage. I 3 is implicit in models using the so–called "riskless hedge" methodology[17]. This methodology equates the return of the riskless asset and the return of a portfolio that consists of a combined option and stock investment. To derive from this statement of equal prices on the portfolio level the price of one unit of the option, one has to divide by the portfolio weights of the option. Exactly at this point the resort to I 3 becomes obvious. – To sum up, neither does the literature find all implications nor does it identify their common root: physical no–arbitrage.

Finally, we want to emphasize that increasing the number of trade objects is not a substitute for physical no–arbitrage. To illustrate this argument look at I 1. One can argue that there are several exchanges, at which an asset can be bought or sold. Yet, instead of demanding the buying and selling prices on all exchanges to coincide – this would be I 1 – one simply regards every purchase/sale combination at several exchanges as a trade object of its own, independent of other trade objects. – This reasoning is not convincing: First, it would be inseparable from an argumentation that considers buying or selling different quantities of the same asset as different trade objects. However, this view would be devastating for arbitrage theory in that it would cause the (sub)linearity of the price functional to collapse. Second, the split–up is a mathematical crutch without economical substance. For, it hides real economic features behind an abstract model setup[18].

By the way, the argumentation does not apply to different prices of buying and selling assets (bid–ask–spread). As buying and selling in such a case are mutually exclusive, purchase and sale are two different trade objects and physical no–arbitrage cannot substitute for this non–artificial split–up.

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[13]See Black/Scholes (1973), p. 643.

[14]See Wilhelm (1981), p. 894.

[15]See Ross (1978), p. 457.

[16]See Harrison/Pliska (1981), p. 226.

[17]See, for example, Ingersoll (1987), p. 313 for an overview.

[18]Although it is correct with respect to the no–arbitrage result as can be seen from the following example: Consider buying and selling one type of asset at two stock exchanges. This yields four "newly created" assets: purchase at exchange 1, sale at exchange 1, purchase at exchange 1, sale at exchange 2, purchase at exchange 2, sale at exchange 1, and purchase at exchange 2, sale at exchange 2. A "purchasing" arbitrage will immediately be possible if an investor decides to sell at exchange 1 and the purchasing prices at exchange 1 and 2 diverge. The same is true for a purchase at exchange 1 and a sale at exchange 1 or 2 ("selling" arbitrage). Finally, to avoid the "classical" arbitrage of simultaneously buying and selling one asset at different prices, purchasing and sale price have to coincide. This means, due to no–arbitrage the prices of all four assets must be the same.

2.2.2 Economical no–arbitrage

Contrary to physical no–arbitrage, economical no–arbitrage connects payoffs to prices[19] rather than securities to prices and thus is **payoff-oriented**:

Economical no–arbitrage means according to the literature the absence of a difference arbitrage. A difference arbitrage (in the most general sense) is a portfolio strategy (possibly combined with cash), that produces (with positive probability) at some time a cash inflow without requiring, at any time, a compensating cash outflow. Formally:

$$\sum_{i=0}^{n} Z_i N_i \geq 0 \ \text{ and } \ \sum_{i=0}^{n} P_i N_i \leq 0 \ \text{ imply } \ \sum_{i=0}^{n} Z_i N_i = 0 \ \text{ and } \ \sum_{i=0}^{n} P_i N_i = 0^{20} \tag{2.2}$$

Of course, condition (2.2) restricted to admissible portfolios should hold in arbitrage-free markets, but it is not a sufficient one in the presence of short sale constraints. For example, a market can in fact be free of arbitrage, while violating (2.2) by means of an inadmissible portfolio composition (see market I). The implications of (2.2) are to weak in such an environment.

Putting these arguments together, we will propose a new definition of economical no–arbitrage. To prepare for the definition, we introduce the concept of actively traded portfolios first[21]:

An admissible portfolio N is said to be "actively traded" if there does not exist an admissible portfolio $\mathbf{N}^*$ with $\mathbf{N}^{*T}\mathbf{Z} \geq \mathbf{N}^T\mathbf{Z}$ and $\mathbf{N}^{*T}\mathbf{P} < \mathbf{N}^T\mathbf{P}$.

The phrase "actively traded" rests on dominance relations between portfolios: Purchasing portfolios dominated by another one cannot be reasonable. Hence, rational non–satiated decision[22] makers will at best supply, but not demand such dominated portfolios. Dominated portfolios are therefore not actively traded.

[19] Which means that economical no–arbitrage implicitly assumes physical no–arbitrage to be given.

[20] First, recognize that statements like $\sum_{i=0}^{n} Z_i N_i \geq 0$ are throughout meant to hold almost surely. Then, the formalization of the no–difference–arbitrage condition will become clear if we recall that $\sum_{i=0}^{n} Z_i N_i > 0$ together with $\sum_{i=0}^{n} P_i N_i = 0$ contains a "free" cash inflow in some states at $t = 1$, $\sum_{i=0}^{n} Z_i N_i = 0$ together with $\sum_{i=0}^{n} P_i N_i < 0$ a "free" cash inflow at $t = 0$, and $\sum_{i=0}^{n} Z_i N_i > 0$ together with $\sum_{i=0}^{n} P_i N_i < 0$ a "free" cash inflow both at $t = 0$ and in some states at $t = 1$.

[21] To simplify notation, we switch to vector notation: $\mathbf{Z}$ denotes an $(n+1) \times 1$ vector of (risky) cash flows, $\mathbf{P}$ the corresponding vector of prices.

[22] See Merton (1973), p. 143.

Equipped with this concept, we can define economical no–arbitrage:

It is impossible to obtain payoffs of actively traded portfolios
and therefore of any admissible portfolio
at an arbitrary low price.

which reads, technically, as:

The portfolio $\mathbf{N}^* = \mathbf{0}$ is actively traded. (2.3a)

To **any** admissible portfolio $\mathbf{N}$ there is an actively traded portfolio $\mathbf{N}^*$ such that $\mathbf{N}^T\mathbf{Z} \geq \mathbf{N}^{*T}\mathbf{Z}$ holds. (2.3b)

Let $\mathbf{N}$ be an admissible and $\mathbf{N}^*$ be an actively traded portfolio. If $\mathbf{N}$ offers higher payoffs in some states and never less than $\mathbf{N}^*$ in all other states, then $\mathbf{N}^T\mathbf{P} > \mathbf{N}^{*T}\mathbf{P}$ must be true. (2.3c)

The interpretation of these technical conditions can be given by a thought–experiment: Look at (2.3a); if $\mathbf{N}^* = \mathbf{0}$ were not actively traded, then there would be an actively traded portfolio $\mathbf{N}$ dominating $\mathbf{N}^*$ by $\mathbf{N}^T\mathbf{P} < 0$ and $\mathbf{N}^T\mathbf{Z} \geq 0$ because payoff and price of $\mathbf{N}^* = \mathbf{0}$ equal zero. Enlarging the quantities held of $\mathbf{N}$ by $z \cdot \mathbf{N}(z \to \infty)$ will almost surely make the investor arbitrarily rich. Condition (2.3c) rules out that dominated payoffs have equal or higher prices than dominating payoffs. Since portfolio $\mathbf{N}$ offers a higher payoff, its price has also to be higher to keep portfolio $\mathbf{N}^*$ actively traded. Otherwise, every non–satiated investor would exploit this dominance relationship. Finally, analyze the reverse scenario, that is imagine (2.3b) is not true (the admissible portfolio $\mathbf{N}$ offers less payoff than $\mathbf{N}^*$) and that the price of portfolio $\mathbf{N}$ is equal or higher than the one of portfolio $\mathbf{N}^*$. $\mathbf{N}^*$ obviously dominates $\mathbf{N}$. For that reason, $\mathbf{N}$ will be in supply, but not in demand as the price charged for it is "too high". Either will the portfolios be withdrawn from the market as a tradable object or the supplier will lower the price as he learned from market's reaction that the price was "too high". Therefore, the assumed price relation demands (2.3b).

To illustrate our no–arbitrage condition beyond clarifying the technical conditions, begin with the phrase "at a low price". This phrase incorporates the ideas of **arbitration of exchange**. Arbitration of exchange stems from the traditional institutionally based literature. It rests on the economic principle and describes nothing else than the desire of getting a given payoff at the minimum price[23] possible.

Modern financial theory has not discussed explicitly this form of an arbitrage because in perfect markets (markets without short selling constraints) the difference arbitrage is the more general concept. Not only does it aim at a given payoff, but also can cover arbitrary payoffs. However, two scenarios can make arbitration of exchange the superior concept: The first one is the concept of an infinite number of assets. Due to the more

[23] For a formalization, see Wilhelm (1987), p. 1163.

complex topological structure in infinite dimensional spaces asset, absence of a difference arbitrage has to be replaced by absence of a free lunch as *Kreps* (1981)[24] points out. Since a free lunch is the possibility of getting arbitrary close to a consumption bundle that is regarded as good at an arbitrarily small cost, free lunches come very near to arbitration of exchange owing to their focus on a given payoff. The second scenario is the presence of market frictions. Under market frictions not replication, but super–replication, that is the idea that no investor will pay more for the financial position than the price of a portfolio that generates at minimum cost at least the same payoff, proved to be optimal. Of course, this super–replication price is nothing else than arbitration of exchange[25].

The other integral part of our economical no–arbitrage condition is the phrase "actively traded". We now know that dominated portfolios are not actively traded. The notion of an actively traded portfolio, however, neither means that there can only be one single actively traded portfolio in the market – every not dominated portfolio is an actively traded portfolio – nor that an actively traded portfolio has to exist at all. For, difference arbitrage means that buying multiples of the cheap asset and selling multiples of the expensive asset has a higher payoff than the assets themselves, which makes both the assets and finite multiples of them not actively traded. Only an additional no–arbitrage condition can assure the existence of actively traded portfolios. And this additional no–arbitrage requirement can be found in the phrase: "not at an arbitrarily low price".

The literature has not yet introduced the concept of actively traded portfolios; we just find some vague hints: *Detemple/Murthy* (1997)[26] and *Schaefer* (1982)[27] observe that there may be dominated portfolios although the market is free of arbitrage (due to short selling constraints). However, they do not translate their observation into a fully–fledged no–arbitrage condition.

To prevent misunderstandings with respect to physical and economical no–arbitrage and especially referring to actively traded portfolios, we finally should clarify four more aspects: First, we have to differentiate actively traded portfolios from so–called non–traded assets. Non–traded assets are not acquired for investment purposes[28]. For example companies purchase raw material to use it in their production process and not to speculate with it. Second, it does make a difference whether an asset is not actively traded or simply non–existent. An asset not actively traded can become an actively traded one after an adequate price adaptation and can exert price influence on other assets due to this opportunity. That feed–back yet will be excluded from the outset if the asset does not exist in the market at all. Third, not actively traded portfolios, that is no investor demands that portfolio under any circumstances, should not be mixed up with a situation, where some investors with a specific desired payoff do not demand an actively traded portfolio. Nevertheless, such an actively traded portfolio principally is worth investing in

[24]See Kreps (1981), pp. 22.

[25]Although even the most recent papers of mathematical finance, see for example Koehl/Pham (2000), p. 343, are not aware of the fact that arguing with the super–replication price in fact means relying on the traditional notion of arbitration of exchange.

[26]See Detemple/Murthy (1997), p. 1157.

[27]See Schaefer (1982), p. 172.

[28]See Hull (1997), p. 288.

and can be held in positive numbers by another investor. Fourth, we have to elucidate whether we are allowed to separate the analysis of physical from the one of economical no–arbitrage. Since economical no–arbitrage works with "the" price of a portfolio and thus relies on physical no–arbitrage, a separation might turn out to be wrong. However, both concepts focus on different aspects of "the" asset price: Physical no–arbitrage deals with assets that are identical from a legal point of view and economical no–arbitrage focuses on economically identical assets. Therefore, both notions of no–arbitrage are indeed separable[29].

3 Solutions to the problems originally posed

Equipped with this new definition of no–arbitrage, we are ready to solve the problems initially posed.

3.1 Solution to problem I: the assumption of "the" price of an asset

We know that physical no–arbitrage (2.2) gives an exact description of the facts and the implications making the circumstances of a transaction (numbers purchased or sold, numbers of exchanges the asset is listed et cetera) irrelevant without misspecifying investors' opportunity set.

With this answer to the question of what the exact conditions are under which "the" price of an asset exists irrespective of the transactions' circumstances, we reconsider our market I: The single price of each asset does not restrict the possibility of trading at different exchanges since the prices at all exchanges must coincide due to physical no–arbitrage. For that reason, using "the" price of assets specifies investors' opportunity sets fully and correctly.

3.2 Solution to problem II: formulation of no–arbitrage, price reproduction, and positivity of the pricing rule under short selling constraints

According to *Wilhelm* (1987)[30] and *Jouini/Kallal* (1995a)[31] we know that

- under short selling there exists a positive and sublinear functional ϕ defined on all payoffs attainable by portfolios such that $\phi(\mathbf{N}^T\mathbf{Z}) \leq \mathbf{N}^T\mathbf{P}$ holds for any portfolio $\mathbf{N}$.

[29] For the same reason it does not make sense to introduce the notion of actively traded portfolios together with physical no–arbitrage.
[30] See Wilhelm (1987), p. 1163.
[31] See Jouini/Kallal (1995a), p. 203.

We therefore do not need to repeat this proof in the text. Instead, we want to point out that our no–arbitrage definition (2.2) and (2.3), although it significantly deviates from the conditions found in the literature, is able to reproduces this result[32].

However, our no–arbitrage condition is capable of doing more than simply to offer an economic foundation of the above result. To see this, we formulate the following advanced consequences of no–arbitrage[33]:

- Under (binding or not) short selling there exists a positive and sublinear functional ϕ defined on all payoffs attainable by portfolios[34] such that $\phi(\mathbf{N}^T\mathbf{Z}) \leq \mathbf{N}^T\mathbf{P}$. This functional delivers
  + for any not actively traded portfolio a lower price bound, that is $\phi(\mathbf{N}^T\mathbf{Z}) < \mathbf{N}^T\mathbf{P}$.
  + for any actively traded portfolio its price, that is $\phi(\mathbf{N}^T\mathbf{Z}) = \mathbf{N}^T\mathbf{P}$. For those portfolios the functional ϕ has the feature of positivity, and price reproduction, but is not necessarily linear and therefore **cannot** be interpreted as be given by state prices.
- In discrete state space[35] for all portfolios not subject to short selling the functional
  + is linear implying $\phi(-\mathbf{N}^T\mathbf{Z}) = -\phi(\mathbf{N}^T\mathbf{Z})$. For those portfolios, ϕ has the feature of positivity, linearity, and price reproduction and can therefore be interpreted as given by (not necessarily unique) state prices.
  + defines a price that is not above the price under (binding or not) short selling constraints.

The intuition behind these statements is as follows: Whenever a portfolio is actively traded, its payoff–price–relation must be fair. Investors seeking to obtain the payoff of an actively traded portfolio at the minimum price possible therefore can do no better than to buy the portfolio itself. Since the price functional is defined as the super–replication price of a given payoff (see arbitration of exchange), we get exactly this price out of the price functional and the price functional reproduces the price of actively traded portfolios. On the other hand, not actively traded portfolios are distinguished by an unfair payoff–price–relation, their price must be too high. The price functional thus determines a lower price bound from which the not actively traded portfolio switches its status from not actively traded to actively traded.

Although the price functional must be positive – a positive payoff cannot have a negative price or a price of zero on arbitrage-free markets – it is not necessarily linear due to (binding or not) short selling constraints. To see this, we illustrate first that the price of a portfolio without short selling constraints, must not lie above the price of the same

[32] See the appendix for a proof.

[33] A formal proof can be found in the appendix.

[34] A technical definition of "attainability by portfolios" can be found in the appendix.

[35] For infinitely dimensional state spaces, it would be necessary to recur to a condition like the absence of "free lunches"; see Kreps (1981), pp. 22.

portfolio subject to (binding or not) short selling constraints: Bear in mind that short selling produces a cash inflow and that this cash inflow lowers the cost of obtaining the desired payoff. Prohibiting short selling excludes this price reducing possibility. As an extreme example consider market I as portrayed in table 1: Without short selling constraints for asset 0 there would be an arbitrage in the market yielding an infinite gain at $t = 0$, that is an infinite negative price. Based on this price relation between portfolios with (binding or not) and without short selling constraints and the fact two individual portfolios $\mathbf{N}^*$ and $\hat{\mathbf{N}}$ may contain inadmissible quantities (for example short selling positions within portfolio $\hat{\mathbf{N}}$), which may disappear by combining $\mathbf{N}^* + \hat{\mathbf{N}}$ (for example because positive quantities in $\mathbf{N}^*$ compensate the negative quantities in $\hat{\mathbf{N}}$), the sublinearity of the price functional becomes clear: Under (binding or not) short selling constraints the sum of the prices of two portfolios $\mathbf{N}^*$ and $\hat{\mathbf{N}}$ when traded separately must be higher than the price of the portfolio $\mathbf{N}^* + \hat{\mathbf{N}}$, that is $\phi(Z^* + \hat{Z}) \leq \phi(Z^*) + \phi(\hat{Z})$ holds.

What still remains to do, is to illustrate why there is a linear price functional for portfolios not subject to short selling constraints. Take into account that any portfolio not subject to short selling constraints is actively traded and its payoff–price–relation must be fair. To see this, assume $\mathbf{N}, -\mathbf{N}$ are admissible and suppose that $\mathbf{N}$ is not actively traded. Then, by definition, there must exists a portfolio $\mathbf{N}^*$ which dominates $\mathbf{N}$. The portfolio $\mathbf{N}^* - \mathbf{N}$, on the other hand, makes $\mathbf{N} = 0$ not actively traded, which is in contradiction to (2.3a). Since for those portfolios buying a negative number and selling the portfolio are identical "trades" and are admissible, both transactions must have the same price, from what a linear price functional follows.

Finally, we want to emphasize that under (binding or not) short selling constraints an universally valid interpretation of the price functional with the help of state prices becomes impossible. Since the definition of a state price demands that the price of a portfolio equals the present value of future payoffs, only a positive **linear** price functional can be interpreted as state prices. Therefore, the state price interpretation solely is valid for portfolios not subject to (binding or not) short selling constraints.

To offer further insights into our advanced consequences of no–arbitrage, contrast them with the literature: *Garman/Ohlson* (1981), *Wilhelm* (1987), who transfers *Garman/Ohlson's* (1981) results to an infinite–dimensional state space, and *Jouini/Kallal* (1995a) are merely able to derive a concrete price of a given payoff by solving explicitly for the super–replication price. Even that part of the literature, that derives price bounds for assets which can be (super)replicated under short selling constraints[36], does not improve our knowledge with respect to the core of the problem: It focuses on an additional asset in the market, but does not offer more pricing information with respect to the original assets in the market. We, on the other hand, are able to propose more precise pricing information without having to solve for super–replication prices. For, we are able to identify a subset of portfolios – not actively traded portfolios – that never, that is independent of investors' payoff preferences, will be priced by the price functional. Moreover, we extend *Jouini/Kallal* (1995a) to partially imperfect markets, that is markets where some,

[36]See Jouini/Kallal (1995a), pp. 204-205, Koehl/Pham (2000), p. 347 for a general no–arbitrage framework and for investors with logarithmic utility function, Detemple/Murthy (1997), p. 1157.

but not all portfolios are subject to short selling constraints, by demonstrating that the outcomes of frictionless markets[37] apply to the subset of portfolios not subject to short selling constraints.

With this answer to the formulation of no–arbitrage, price reproduction, and positivity of the price functional under short selling constraints, we reconsider our market I: A portfolio made of $1\frac{1}{3}$ units of asset A_j dominates asset 0[38]; due to its short selling constraint asset A_0 is not actively traded. Asset A_i and A_j do not dominate each other and there is no arbitrage in the market. Since both assets are not subject to (binding or not) short selling constraints and therefore actively traded, we know that a linear price functional ϕ prices both assets exactly. Using asset A_i and asset A_j to calculate the price functional, we obtain $\phi_{S_1} = \dfrac{3}{8}$ and $\phi_{S_2} = \dfrac{3}{8}$. As this price functional is positive, linear, and has the feature of reproducing the prices of all actively traded portfolios in the market, it can be interpreted as state prices. This price functional in addition allows us to derive the lower price bound for the not actively traded asset A_0: The price, at which it switches its status from "not actively traded" to "actively traded", is 75.

We want to close this subsection with two final remarks. First, we have seen that mistakes in the course of determining the price functional can happen quite easily, namely when one calculates the price functional out of portfolios subject to (binding or not) short selling constraints. That is exactly what happened in our problem II: The not actively traded asset A_0 was falsely used to compute the price functional and even state prices. Therefore, the status "subject to short selling constraints or not" is an additional information, one needs besides portfolios' prices and payoffs to compute a generally valid price functional. Second, a positive linear price functional does not necessarily imply that all state prices have to be positive. It just demands that a positive payoff must have a positive price. We will return to this point when we will discuss the CAPM in subsection 3.4.2.

3.3 Solution to problem III: the endogenous justification of a nonnegative riskless rate

In section 3.1 we found that working with "the" price of an asset solely depicts investors' opportunity sets correctly when we recur to the idea of physical no–arbitrage. Justifying endogenously a nonnegative riskless rate again ends up in an examination of investors' opportunity sets.

The mainstream literature assumes that investors are able to invest in a riskless and several risky assets. Yet, this is an incorrect description of their opportunity sets because one asset is missing: cash. Cash is an "asset" that is always available. Every amount of money that is neither consumed nor invested (in the strict sense) today is automatically transferred to tomorrow. Due to this automatism, cash seems to be invisible, but nevertheless belongs

[37] See for example Dybvig/Ross (1992).

[38] This portfolio offers at the same price of 100 a payoff of $106\frac{2}{3}$ in the first and 160 in the second state. Alternatively, one can argue asset i and j synthesize a riskless asset, whose rate of $33\frac{1}{3}\%$ dominates asset A_0's rate of 0%.

to investors' investment opportunity sets.

Using this more adequate description of investors' investment opportunity sets, we know that a positive riskless rate dominates cash and cash hence will not be actively traded. A negative riskless rate, however, is dominated by the zero "interest" rate of cash. As a consequence, a riskless asset with a negative riskless rate is not actively traded.

To be more precise, look at the following formalization of the above intuition: If there is an actively traded riskless asset A_a with rate r, its price is equal to its discounted payoff, by definition and by no–arbitrage:

$$P_a = \phi(1) = \frac{1}{1+r} \tag{3.1a}$$

Since cash offers a payoff of $Z_0 = 1$ for an investment of $I_0 = 1$, a positive riskless rate $r > 0$ dominates cash and makes it not actively traded. Therefore, we arrive at a lower price bound for cash:

$$\frac{1}{1+r} = \phi(1) \leq 1 = I_0 \tag{3.2a}$$

On the other hand, a negative riskless rate $r < 0$ would result in cash to be actively traded and asset A_a with investment I_a not actively traded with consequences

$$P_0 = \phi(1) = 1 \tag{3.1b}$$

and a lower price boundary I_a for the investment in A_a

$$\phi(1) = 1 \leq \frac{1}{1+r} = I_a \tag{3.2b}$$

Or to put it differently: We now know that the "reactivation" of the "invisible" asset cash in combination with the notion of actively traded portfolios offers a endogenous justification[39] of a nonnegative riskless rate.

Although cash is a quite common asset, the literature has ignored its consequences on asset pricing. We are only aware of one source, a textbook for students, that introduced cash in form of an example[40] into arbitrage theory. But a systematic integration into arbitrage theory is of course beyond the scope of a textbook like that.

With the knowledge of this section, reconsider our market II thereby tackling the problem of an endogenous justification of a nonnegative riskless rate: Market II is not been correctly specified because it ignores cash. Cash dominates A_a and a portfolio composed of 1.7 units cash and –0.75 units A_j dominate A_i. Both A_a and A_i hence are not actively traded. In fact, market II is not even free of arbitrage. To guarantee no–arbitrage, we have to impose short selling constraints on A_a and A_i.

[39] We do not claim, however, to give a justification of why cash or other assets exist at all; their existence is exogenous in our model. Only the consequences from their existence are topic of our analysis.

[40] See Schäfer/Kruschwitz/Schwake (1998), p. 131.

The riskless rate implied by this market can now be found by determining the super–replication price of a payoff x in both state, where x denotes an arbitrary positive number. The solution to this problem on market II reads $\phi\begin{pmatrix} Z(S_1)=x \\ Z(S_2)=x \end{pmatrix} = x$. Therefore, the riskless rate is nonnegative ($r = 0$).

Moreover, we are ready to comment on the assertion that it is possible to assure a positive riskless rate by working with several riskless assets since one asset with a positive rate automatically forces the other riskless rates to be positive. Our example first clarifies where more than one riskless assets should come from; we have one explicit (A_a) and one implicit riskless asset (constructed out of A_i and A_j). Second, it shows that, although there are two riskless investment opportunities, we nevertheless can have a negative riskless rate. Therefore, arguing with several riskless assets is a complicated and not necessarily successful way to assure a nonnegative riskless rate. We definitely need cash.

Closely related to the question of a nonnegative riskless rate is another problem: the position of the borrowing rate.

In addition to the riskless asset A_a with rate r consider a borrowing opportunity A_s at rate r_s with payoff $Z_s = 1$ and initial cash inflow $I_s = \frac{1}{1+r_s}$. Lending and borrowing are subject to short selling constraints in that lending is only allowed in positive, borrowing in negative quantities (emission of assets!). If lending and borrowing are to be both actively traded, no–arbitrage will demand (for 1 unit lending and –1 unit borrowing)

$$\phi(1) = \frac{1}{1+r} \quad \text{and} \quad \phi(-1) = -\frac{1}{1+r_s} \tag{3.3}$$

Due to the feature of sublinearity of the price functional, we get

$$\phi(0) = \phi\Big(1+(-1)\Big) \leq \phi(1) + \phi(-1) \tag{3.4}$$

and hence

$$\phi(0) = 0 \leq \frac{1}{1+r} + \left(-\frac{1}{1+r_s}\right) \tag{3.5}$$

that is $r_s \geq r$: The borrowing rate cannot lie below the lending rate, which in turn has to be nonnegative[41].

[41] A generalization of this problem, namely that the purchasing price cannot be located below the selling price of an asset, can be found in Jouini/Kallal (1995b), p. 179. However, since they do not focus on interest rates, they do not derive the nonnegativity condition on both interest rates, which was our goal.

3.4 Solution to problem IV: the connection between arbitrage theory and special pricing theories

For the rest of the paper we concentrate on these cases where short selling constraints are absent, every portfolio is, hence, actively traded, and there is only a finite number of states[42]. By constituting the structure of fair asset prices for those portfolios,

$$P_i = \sum_{s \in \mathcal{S}} \phi_s Z_{is} \tag{3.6}$$

$\mathcal{S}$ set of possible states

s single state from $\mathcal{S}$

arbitrage theory undoubtedly is a pricing theory albeit a quite universal one. Arbitrage theory, however, does not compute state prices because it does not solve equation (3.6) for ϕ_s; this task is fulfilled by special pricing theories: law–of–one–price–oriented and utility–based pricing[43], which thereby discuss the economic forces behind the state prices.

3.4.1 Law–of–one–price–oriented pricing of derivatives

3.4.1.1 A brief digression on the law of one price [44]

The law of one price – it was originally due to *Jevons* (1871) and rediscovered by *Ross* (1978) for financial markets – describes an important fact: Economically identical goods must have identical prices independent of investors' preferences. To be more precise, we can distinguish between two variants[45]: The first variant demands two portfolios $\mathbf{N}$ and $\mathbf{N}^*$, whose payoff coincide in all states at $t = 1$, to have the same price at $t = 0$. The second variant focuses on special portfolios, so–called arbitrage portfolios. Arbitrage portfolios offer in every state at $t = 1$ a payoff of zero and must have a price of zero. Both variants are equivalent in the absence of short selling constraints: That the first variant implies the second one is obvious because the second one studies not an arbitrary payoff, but a payoff of zero. The reverse implication results as follows: Forming the difference between the payoffs of economically identical goods yields a total payoff of zero. This payoff must have a price of zero proving that the second variant can be derived from the first one.

[42] As we do not consider continuous trading, this assumption is necessary to be able to span payoffs even in a one–period model.

[43] The identification of these basic forms of the numerous variants of special pricing theories is the merit of Schöbel (1995).

[44] For related issues on the law of one price, consult Oh/Ross/Shanken (1998)

[45] See Ingersoll (1987), p. 59.

3.4.1.2 Translation of the law of one price into pricing

The typical application of law–of–one–price–oriented pricing constructs portfolios composed of the underlying (stock) of the derivative ($A_1, \ldots, A_{n-1}$), the riskless asset (A_a), and the derivative (A_n)[46] so that the portfolios' payoffs coincide at $t = 1$ in all states. Hence, (law of one price!) at $t = 0$ all portfolios must have the same price, enabling a preference–free computation of the desired price of the derivative.

Analyzing the huge economic literature on derivative pricing, we can extract three basic principles of law–of–one–price–oriented pricing to construct portfolios and therefore three different types of portfolios: duplication portfolio, hedge portfolio, and arbitrage portfolio:

1. Duplication portfolio:

   This form of the law–of–one–price–oriented pricing searches for the quantities of stocks and the riskless asset which synthesize exactly (duplicate) the payoff of the derivative in all states $s \in \mathcal{S}$ at $t - 1$, that is

$$Z_{ns} = N_a + \sum_{i=1,\, i\neq a}^{n-1} N_i \cdot Z_{is} \qquad \text{for all} \quad s \in \mathcal{S} \tag{3.7}$$

2. Hedge portfolio:

   The hedge portfolio combines stocks and derivative that way that a riskless payoff results in all states $s \in \mathcal{S}$ at $t = 1$, that is

$$-N_a = \sum_{i=1,\, i\neq a}^{n-1} N_i \cdot Z_{is} - Z_{ns} \qquad \text{for all} \quad s \in \mathcal{S} \tag{3.8}$$

   or rather

$$1 = -\sum_{i=1,\, i\neq a}^{n-1} \frac{N_i}{N_a} \cdot Z_{is} + \frac{Z_{ns}}{N_a} \qquad \text{for all} \quad s \in \mathcal{S} \tag{3.9}$$

3. Arbitrage portfolio:

   Based on *Merton* (1973), the arbitrage portfolio puts together stocks, riskless asset, and derivative in such a manner that there is at $t = 1$ with certainty, that is in all states $s \in \mathcal{S}$, a riskless payoff of zero.

$$0 = N_a + \sum_{i=1,\, i\neq a}^{n-1} N_i \cdot Z_{is} - Z_{ns} \qquad \text{for all } s \in \mathcal{S} \tag{3.10}$$

[46] For simplicity, we assume that there is just one derivative. Generalizations are of course possible, but do not deliver any further results.

The above systems all have as many equations as there are states in the economy. If an arbitrary derivative shall be priced, obviously the number of assets has to be at least as large as the number of states. This fact that the payoff of a derivative can be written as a linear combination of payoffs of stocks and riskless asset is called **spanning property**. Spanning thus implies – expressed with the help of equation (3.6) – the possibility of depicting the state prices necessary for pricing by real assets.

Law–of–one–price–oriented pricing directly recurs to the implications (duplication portfolio)

$$Z_{ns} = N_a + \sum_{i=1,\, i\neq a}^{n-1} N_i \cdot Z_{is} \ \text{ for all } s \in \mathcal{S} \ \Rightarrow \ P_n = \frac{N_a}{1+r} + \sum_{i=1,\, i\neq a}^{n-1} N_i \cdot P_i \tag{3.11}$$

or rather equivalent formulations like (hedge portfolio)

$$1 = -\sum_{i=1,\, i\neq a}^{n-1} \frac{N_i}{N_a} \cdot Z_{is} + \frac{Z_{ns}}{N_a} \ \text{ for all } s \in \mathcal{S} \ \Rightarrow \ \frac{1}{1+r} = -\sum_{i=1,\, i\neq a}^{n-1} \frac{N_i}{N_a} \cdot P_i + \frac{P_n}{N_a} \tag{3.12}$$

or (arbitrage portfolio)

$$0 = N_a + \sum_{i=1,\, i\neq a}^{n-1} N_i \cdot Z_{is} - Z_{ns} \ \text{ for all } s \in \mathcal{S} \ \Rightarrow \ 0 = \frac{N_a}{1+r} + \sum_{i=1,\, i\neq a}^{n-1} N_i \cdot P_i - P_n \tag{3.13}$$

By directly using the payoff/price equivalence of two portfolios in the form "identical payoff $\Rightarrow$ identical price" in equation (3.7) and (3.11) or rather (3.8) and (3.12) (first variant of the law of one price) or rather "payoff of zero $\Rightarrow$ price of zero" in (3.10) and (3.13) (second variant of the law of one price), the name "law–of–one–price–oriented" pricing becomes clear. However, with respect to equation (3.12) one remark is in order: Hedge portfolios do not calculate the price of the derivative, but $\dfrac{P_n}{N_a}$. The notion of physical no–arbitrage thus is needed to draw conclusions on the price per unit of the derivative similar to the other forms of law–of–one–price–oriented pricing[47].

To show the connection between arbitrage theory and law–of–one–price–oriented pricing, observe that the law–of–one–price–oriented pricing works without explicit knowledge of ϕ although it can calculate ϕ from prices[48]. Should spanning of the derivative payoff fail however, every derivative price would be compatible with the law of one price (although not with the no–arbitrage conditions (2.2) or rather (2.3)) and the unambiguous price determination fails. Therefore, law–of–one–price–oriented pricing merely has the feature of price reproduction and linearity for a subset of portfolios. Proving that the

[47] That the law of one price indeed is unable to care of this aspect can easily be demonstrated with the help of market I: With a price of assets A_i and A_k of 75, we have as fair price of 10 units of asset A_j 750. Should, however, the price of one unit of asset A_j be unequal to $\frac{750}{10}$, the physical no–arbitrage condition (2.2) is violated.

[48] See equations (3.11) to (3.13).

state prices from law–of–one–price–oriented-pricing at least specify the price functional of arbitrage theory for this subset, we have to check positivity: Since under missing short selling constraints all portfolios are actively traded on arbitrage–free markets, there cannot be a dominating payoff, especially no positive payoff with a negative price. This argument in turn keeps the price functional positive. For illustrative purposes consider the *Cox/Ingersoll/Ross* (1979) binomial model: To have both the stock and the riskless asset to be actively traded, implies a positive price of the stock and $u > r > d$ for the stock's return, that it exactly the conditions *Cox/Ingersoll/Ross* (1979) employ[49] to guarantee positive state prices. As the price functional calculated by means of the law–of–one–price–oriented–pricing reproduces asset prices and is linear as well as positive, law–of–one–price–oriented-pricing indeed specifies the price functional (3.6) (and the state prices) of arbitrage theory.

That way, we generalize *Cox/Ross/Rubinstein* (1979), who show this connection between law–of–one–price–oriented pricing and arbitrage theory just for the binomial model.

In closing this section – for concrete details on law–of–one–price–oriented pricing, we refer to the well–known literature –, we want to address a semantic problem. The literature sometimes[50] uses the somewhat misleading phrase "arbitrage–oriented pricing" instead of "law–of-one–price–oriented pricing". This concept formation stems form the consideration that two portfolios with identical payoffs must have – due to arbitrage reasons - an identical price independent of investors' preferences. Our exposition in the foregoing section yet identified this reasoning as law-of-one-price and thus just a part of arbitrage theory. Since utility-based pricing of section 3.4.2 also relies on arbitrage theory, we prefer a formulation that exactly names the aspect of arbitrage theory used, namely law–of–one–price–oriented pricing.

3.4.1.3 Law of one price versus completeness of markets

Crucial for law–of–one–price–oriented pricing was the spanning property, which must not get mixed up with completeness of markets. Completeness means that **all** state prices of the market can be portrayed with the help of real assets and consequently results in a price functional for all asset in the market independent of investors' preferences. This connection, which is stated in a technical way as the equivalence of an unique martingale measure and completeness of markets[51], is called **second fundamental theorem of asset pricing**. Spanning and completeness imply insofar different ideas on the universal validity of their pricing statements. Whereas spanning aims at one concrete derivative, completeness prices all assets in the market. Law–of–one–price–oriented pricing, however, just needs spanning to determine prices of derivatives and definitely not unique state prices and hence the second fundamental theorem of asset pricing.

To be able to illustrate the difference between spanning and completeness, we focus on an extended version of a market I – cash is a not actively traded asset and therefore can

[49] See Cox/Ingersoll/Ross (1979), p. 240.
[50] See for example Schöbel (1995).
[51] See Harrison/Pliska (1983), p. 314.

be skipped without restricting investors' opportunity set:

		states of the world			
asset	price	state 1	state 2	state 3	state 4
A_a	100	110	110	110	110
A_i	100	110	90	110	140
A_j	100	120	80	80	120
A_{CO} (call option on A_j	?	10	0	0	10

Table 4: payoffs and prices on an extended market I

The unknown price of the option can be obtained from spanning its payoff by means of a portfolio constructed of A_j and the riskless asset A_a, that is we have:

state 1: $120 \cdot N_j + 110 \cdot N_a = 10$

state 2: $80 \cdot N_j + 110 \cdot N_a = 0$

state 3: $80 \cdot N_j + 110 \cdot N_a = 0$

state 4: $120 \cdot N_j + 110 \cdot N_a = 10$

The duplications portfolio reads $N_j = \frac{1}{4}$ as well as $N_a = -\frac{2}{11}$ leading, according the law of one price, to an option price of $P_{CO} = \frac{1}{4} \cdot 100 - \frac{2}{11} \cdot 100 = 6\frac{9}{11}$. Completeness on the other hand, is not primarily interested in the option price, but calls for unique state prices. Since in our example $\phi_{S_1} = -\phi_{S_4} + \frac{15}{22}$, $\phi_{S_2} = \frac{3}{2} \cdot \phi_{S_4}$, and $\phi_{S_3} = -\frac{3}{2} \cdot \phi_{S_4} + \frac{5}{22}$ hold, the prices of states 1 to 3 just can be expressed as a function of the fourth state price and are therefore not unique. The market is incomplete[52].

3.4.2 Utility-oriented pricing

3.4.2.1 Basics

Starting point of utility-based pricing are optimum portfolio decisions of an investor reflecting his individual market price of risk. With the help of this market price of risk, the payoff at $t = 1$ of the derivative can be priced. If the investor concerned is an individual decision maker, the derivative price will be a subjective indifference price; if the investor

[52] And, to reemphasize this point, not actively traded portfolios like cash cannot make it complete.

is representative of a market in equilibrium, we will obtain equilibrium based derivative prices.

Examine a bit closer the underlying decision problem of the investor: A decision maker has initial wealth W_0, that can be invested in stocks $(A_1, \ldots, A_{n-1})$, the riskless asset (A_a), and the derivative (A_n). The risky terminal wealth $\tilde{W}$ of this strategy (random variable with realization W_s in state s) reads as follows:

$$\tilde{W} = \sum_{i=1}^{n-1} \big[Z_i - (1+r)P_i\big] N_i + \big[Z_n - (1+r)P_n\big] N_n + (1+r)W_0 \tag{3.14}$$

The decision makers aims to maximize the expected utility $U[\cdot]$ of terminal wealth by optimally selecting the number of units of stocks $(N_1, \ldots, N_{n-1})$ and the derivative (N_n). Formally:

$$\operatorname*{Max}_{\mathbf{N}} E\{U[\tilde{W}]\} \tag{3.15}$$

Denoting the expected utility of terminal wealth as Φ, the necessary conditions for the optimum numbers read

$$\frac{\partial \Phi}{\partial N_i} = 0 = E\left\{\frac{\partial U[\tilde{W}]}{\partial \tilde{W}} \cdot \frac{\partial \tilde{W}}{\partial N_i}\right\} = E\big\{U'\big[\tilde{W}\big]\big[Z_i - (1+r)P_i\big]\big\} \text{ for } i = 2, 3 \tag{3.16}$$

The desired derivative price P_n is immediately obtained from (3.16) for $i = n$ by solving this equation with respect to P_n (with $E\big\{U'\big[\tilde{W}\big]\big\} \neq 0$ required):

$$P_n = E\left\{\frac{1}{1+r} \cdot \frac{U'[\tilde{W}]}{E\{U'[\tilde{W}]\}} \cdot Z_n\right\} = \sum_{s \in \mathcal{S}} \phi_s \, Z_{ns} \tag{3.17}$$

Two remarks are in order: First, no–arbitrage guarantees that problem (3.15) has indeed a solution, that is that the price system is viable[53]. Second, equation (3.17) contains two possible cases. If the derivative is in zero–net supply, the price of risk calculated from the stocks will be passed on to the derivative. If the derivative, however, is held in non–zero numbers, the price of risk stems from the optimum terminal wealth and cannot be computed just from one asset class. Moreover, in both cases, the type of the utility function and the variety of available stocks obviously exert influence on the price of risk. This means that the introduction of a new stock usually leads to new derivative prices[54].

To show the connection between arbitrage theory and utility–oriented pricing, we have to prove that the term from equation (3.17)

$$\phi_s = \mu_s \frac{1}{1+r} \cdot \frac{U'[\tilde{W}]}{E\{U'[\tilde{W}]\}} \quad \text{for all} \quad s \in \mathcal{S} \tag{3.18}$$

μ_s probability of state s

[53] See Harrison/Kreps (1979), p. 386.

[54] For a more intense discussion of issues like this, see Ingersoll (1987), chapter 2.

indeed is a state price. Obviously, ϕ prices all actively traded portfolios of the market segment (price reproduction and linearity). Since in addition ϕ_s in equation (3.18) turns out to be positive because we assumed strictly positive marginal utility, and $\sum_{s \in \mathcal{S}} \phi_s = \frac{1}{1+r}$ holds, ϕ_s in equation (3.18) must be a specification of the price functional (3.6) of arbitrage theory (universal pricing theory).

So far, we not only reproduced the result of *Duffie* (1996)[55], but also showed how the various variants of utility–oriented pricing fit into this scheme. Now we are ready to analyze connections between law–of–one–price–oriented and utility–oriented pricing. Although utility-oriented pricing seems to deliver completely different prices at first sight, both pricing approaches will coincide if the spanning property with respect to the derivative holds. Formally, due to (3.17) in connection with (3.18) we immediately have congruence of utility–oriented (3.17) with law–of–one–price–oriented (3.11) derivative pricing. Economically, both approaches coincide because a derivative, that can be duplicated, offers independent of investors' utility functions the same utility as the duplication portfolio. Insofar, law–of–one–price–oriented and utility–oriented pricing do not conflict with each other, but simply rely on different restrictive assumptions: Law–of–one–price–oriented pricing rests on the spanning assumption, whereas utility-oriented pricing is founded on the expected utility principle.

This means, we finally found an answer to our problem IV, namely of how arbitrage theory, law–of–one–price–, and utility–oriented pricing are related.

3.4.2.2 CAPM

To close this section, we want to mention a special case of the equilibrium–based variant of utility–oriented pricing: the Capital Asset Pricing Model (CAPM). It is a special case because it relies on restrictive assumptions: On the one hand, it may be based on a quadratic utility function. However, in a world with free disposal, that is the possibility of "throwing away" money at no cost, quadratic utility on the payoffs cannot be rational since negative marginal utility can be avoided at no cost. Quadratic utility hence is not a good device for basing the CAPM on and we will not discuss this line of the CAPM derivation further. On the other hand, the CAPM is often grounded on the assumption of normally distributed payoffs. Normality, however, implies that there is never a portfolio payoff that is positive in every state besides an exclusive investment in the riskless asset. In such a case the positivity of the pricing rule no longer is equivalent to positive state prices. To see this, transform according to *Wilhelm* (1983)[56] the CAPM price equation

[55]See Duffie (1996), pp. 7-8.
[56]See Wilhelm (1983), p. 16.

and apply it to derivative pricing as follows:

$$P_n = E\left\{\frac{1}{1+r}\left[1 - \frac{E\{W_M\} - (1+r)W_{M0}}{\mathrm{var}\{W_M\}}\big(W_M - E\{W_M\}\big)\right]\cdot Z_n\right\} \quad (3.19)$$

W_M	wealth of the market portfolio at $t = 1$
W_{M0}	wealth of the market portfolio at $t = 0$

with state prices[57]:

$$\phi_s = \mu_s \cdot \frac{1}{1+r}\left[1 - \frac{E\{W_M\} - (1+r)W_{M0}}{\mathrm{var}\{W_M\}}\big(W_M - E\{W_M\}\big)\right] \quad (3.20)$$

Owing to equation (3.20), a wealth interval based on the normal distribution can easily lie far above the expected wealth level leading to a negative state price, a point that was originally observed by *Dybvig/Ingersoll* (1982)[58]. Nevertheless, we know that a positive payoff must have a positive price because in the absence of short selling constraints every portfolio is actively traded on arbitrage–free markets. Therefore, the price functional must be positive (as long it is applied to linear combinations of the primary assets in the market). And it is exactly this result that goes beyond *Dybvig/Ingersoll* (1982).

4 Conclusion

We started from the observation that today's mathematical finance does not discuss important economic aspects of pricing. This became evident in four cases: First, the assumption of "the" price of assets and consequences to investors' opportunity set, second, the formulation of no–arbitrage, price reproduction, and positivity of pricing rules under short selling constraints, third, the explicit assumption of a positive riskless rate, and fourth, the missing integration of special pricing theories into arbitrage theory (= universal pricing theory). Our paper has found the following solutions to these problems:

"The" price of an asset will only give an exact description of investors' opportunity sets when the concept of physical no–arbitrage is introduced. The second problem can be solved by relying on a more precise form of an economical no–arbitrage condition. Economical no–arbitrage under short selling constraints means that it is impossible to acquire actively traded and therefore every admissible portfolios at an arbitrary low price. From the above definition of no–arbitrage it follows a price functional that determines for all payoffs a lower price bound. This price functional determines exactly the price of actively portfolios, for not actively traded portfolios it defines a lower price bound. However, under (binding or not) short selling constraints this price functional is not necessarily linear, which means it cannot be interpreted as given by state prices. A linear price functional

[57] Since a normal distribution has an infinite number of states, we have to discretize its state space by defining wealth intervals as states, for example wealth is between 1000 and 1002.

[58] See Dybvig/Ingersoll (1982), p. 238.

and thus state prices only apply to portfolios not subject to short selling constraints. The notion of actively traded portfolios also helped to tackle the third problem. Recall that in every economy there is an "invisible" asset cash, which is always available. Every riskless asset with a negative interest rate is dominated by cash and is therefore not actively traded, an reasoning offering an endogenous justification of a nonnegative riskless rate. Finally, we have been able to show that special pricing theories (law–of–one–price–oriented and utility–oriented pricing) just concretize the relationship between asset prices and their payoffs given by arbitrage theory (= almost universal pricing theory), using different restrictive assumptions: Law–of–one–price–based pricing rests on the spanning assumption, whereas utility–oriented pricing is founded on the expected utility principle.

With these three amendments and the discovery of the connection between arbitrage theory and special pricing theories, we brought back some economic intuition into mathematical finance. What remains to do, however, is to transfer our findings to a multi period, maybe continuous time, framework.

Appendix

- *Proof that implications (I 1) to (I 3) are indeed consequences of physical no–arbitrage:*

The proof will carried out in several steps:

Proposition 1: For every portfolio $\mathbf{N}$, that can be sold short, under all circumstance u $P_u(-\mathbf{N}) = -P_u(\mathbf{N})$ holds.

Proof: This feature follows immediately from the obvious equality $0 = P_u(0)$ in connection with $P_u(\mathbf{0}) = P_{u_0}(\mathbf{N} - \mathbf{N}) = P_{u_1}(\mathbf{N}) + P_{u_2}(-\mathbf{N})$.

Proposition 2: For every natural number z and every portfolio $\mathbf{N}$, $P_u(z \cdot \mathbf{N}) = z \cdot P_u(\mathbf{N})$ holds under all circumstances u.

Proof: This is obvious since $z = 1+1+\ldots+1$ and hence $z \cdot \mathbf{N} = \mathbf{N}+\mathbf{N}+\ldots+\mathbf{N}$ is true.

Corollary 1: If portfolio $\mathbf{N}$ can be sold short, proposition 2 also holds for negative integers.

Proposition 3: For every positive rational number j/k $P_u\left(\frac{j}{k}\mathbf{N}\right) = \frac{j}{k} P_u(\mathbf{N})$ holds under all circumstances u.

Proof: Owing to proposition 2, it suffices to demonstrate this feature for $j = 1$. Assume $\hat{\mathbf{N}} = \frac{1}{k}\mathbf{N}$. Then, we have $\mathbf{N} = k \cdot \hat{\mathbf{N}}$, thus $P_u(\mathbf{N}) = P_u(k \cdot \hat{\mathbf{N}}) = k\, P_u(\hat{\mathbf{N}})$, and hence $P_u\left(\frac{1}{k}\mathbf{N}\right) = P_u(\hat{\mathbf{N}}) = \frac{1}{k} P_u(\mathbf{N})$.

Corollary 2: If portfolio $\mathbf{N}$ can be sold short, proposition 3 also holds for negative rational numbers.

Proposition 4: For every positive real number x, $P_u(x \cdot \mathbf{N}) = x \cdot P_u(\mathbf{N})$ holds under all circumstances u.

Proof: Consider a sequence of rational monotonously increasing positive numbers $\{a_1, a_2, \dots\}$ converging to x. Construct the following sequence of rational numbers: $y_1 = a_1$, $y_2 = a_2 - a_1$, $y_3 = a_3 - a_2, \dots$. Then, due to proposition 3 and the required additivity of quantities $\mathbf{N}_{u_0} = \mathbf{N}_{u_1} + \mathbf{N}_{u_2} + \dots$, it follows:

$$P_{u_0}(x \cdot \mathbf{N}_{u_0}) = P_{u_0}(y_1 \mathbf{N}_{u_1} + y_2 \mathbf{N}_{u_2} + \dots) = \sum_i y_i P_{u_i}(\mathbf{N}_{u_i}) = x\, P_{u_0}(\mathbf{N}_{u_0})$$

Corollary 3: If portfolio $\mathbf{N}$ can be sold short, proposition 4 also holds for negative real numbers.

Summing up, from proposition 1 to 4 we obtain that the price of a portfolio must be equal to the weighted sum of its components, that is (I 2). This case includes a special portfolio, namely buying or selling just one unit of an asset and hence (I 1). Moreover, from proposition 4, (I 3) can be directly obtained.

- *Proof that our no–arbitrage definition indeed yields a sublinear price functional:*

+ Preliminary work:

We call a payoff Z^* "attainable" if there exists an admissible portfolio $\mathbf{N} \in X$ such that $Z^* \leq \mathbf{N}^T\mathbf{Z}$ holds. Such payoffs may be "attained" by buying $\mathbf{N}$ and "throwing away" the quantities $\mathbf{N}^T\mathbf{Z} - Z^* \geq 0$ since there is free disposal.

Proposition 5: The set $M(X)$ of all attainable payoffs is a convex cone with vertex 0.

Proof: Assume $Z^* \leq \mathbf{N}^{*T}\mathbf{N}$ and $\hat{Z} \leq \hat{\mathbf{N}}^T\mathbf{Z}$ with $\mathbf{N}^*, \hat{\mathbf{N}} \in X$. According to our proofs so far, for $\alpha, \beta \in \mathbb{R}^+$ $\alpha\mathbf{N}^* + \beta\hat{\mathbf{N}} \in X$ holds, that is X is a convex cone. But it is clear that $\alpha Z^* + \beta\hat{Z} \leq \alpha\mathbf{N}^{*T}\mathbf{Z} + \beta\hat{\mathbf{N}}^T\mathbf{Z} = \big(\alpha\mathbf{N}^{*T} + \beta\hat{\mathbf{N}}^T\big)\mathbf{Z}$ holds, which proves our assertion.

+ Core of the proof:

Proposition 6: There exists a positive and sublinear functional ϕ on $M(X)$ such that $\phi(\mathbf{N}^T\mathbf{Z}) \leq \mathbf{N}^T\mathbf{P}$ holds for any portfolio $\mathbf{N}$.

Proof: The proof will be carried out in several steps:

1. There is a well–defined price functional ϕ:

According to the notion of arbitration of exchange, ϕ is defined to assign to every attainable payoff at $t = 1$ the price of a portfolio that generates at minimum cost at least the same payoff (super–replication price), that is

$$\phi(Z^*) := \inf\left\{\mathbf{N}^T\mathbf{P}\middle|\mathbf{N} \in X,\, \mathbf{N}^T\mathbf{Z} \geq Z^*\right\}$$

The existence of ϕ immediately follows from (2.3b).

2. Homogeneity of degree 1 of the price functional ϕ:

Begin with $\phi(0) = 0$. This feature follows from assumption (2.3a), namely the fact that portfolio $\mathbf{N} = \mathbf{0}$ (doing nothing) is actively traded and thus not dominated by another alternative.
For the case $\alpha > 0$, we just recall that $X_{\text{new}} = \alpha \cdot X_{\text{old}}$ holds.

3. Subadditivity of the price functional ϕ:

To prove subadditivity $\phi(Z^* + \hat{Z}) \leq \phi(Z^*) + \phi(\hat{Z})$, it suffices to remember that the individual portfolios $\mathbf{N}^*$ and $\hat{\mathbf{N}}$ may contain inadmissible quantities (for example short selling within portfolio $\hat{\mathbf{N}}$), which may disappear by combining $\mathbf{N}^* + \hat{\mathbf{N}}$ (for example because positive quantities in $\mathbf{N}^*$ compensate the negative quantities in $\hat{\mathbf{N}}$).

4. Positivity of the price functional ϕ:

Because of (2.3a) the price of doing nothing equals zero. In addition, ϕ is (strictly) monotone with respect to payoffs of (actively traded) portfolios (see (2.3c)). Now the positivity of ϕ immediately follows from $\phi(0) = 0$.

5. The remaining properties of ϕ, as claimed in the proposition, directly follow from its construction.

- *Some more features of the price functional:*

Proposition 7: There exists a positive and sublinear functional ϕ on $M(X)$ such that $\phi(\mathbf{N}^T\mathbf{Z}) = \mathbf{N}^T\mathbf{P}$ holds for any **actively traded** portfolio $\mathbf{N}^*$.

Proof: To see this, recall that $\phi\big(\mathbf{N}^T\mathbf{Z}\big) = \mathbf{N}^T\mathbf{P}$ is true, by construction, for every actively traded portfolio.

Corollary 4: If $z, -z \in M(X)$ holds (that is the portfolio under consideration can be sold short), then $-\phi(z) = \phi(-z)$.

Proof: Let be $z = \mathbf{N}^T\mathbf{Z}$, $-z = \left(-\mathbf{N}^T\right)\mathbf{Z}$, then $\phi(-z) = \mathbf{N}^T\mathbf{P} = -\left(-\mathbf{N}^T\mathbf{P}\right) = -\phi(z)$ holds from proposition 7 and the fact that portfolios without short selling constraints are actively traded.

For the rest our propositions we assume a finite state space, the number of possible states will be denoted by K, so that $M(X) \subset \mathbb{R}^K$ holds.

Proposition 8: There exists a positive linear functional φ on $\mathbb{R}^K$ such that $\varphi(z) \leq \phi(z)$ holds for any portfolio payoff $z \in M(X)$. In addition, if $z, -z \in M(X)$ holds, that is the portfolio may be sold short, then $\varphi(z) = \phi(z)$ is true[59].

Proof: The proof uses a standard separating hyperplane argument[60]: We define two convex sets in $\mathbb{R}^{K+1}$ by $Y = \left\{(z,x)\middle| z \in M(X), \phi(z) \leq x\right\}$ and $\overline{Y} = \left\{(z,0)\middle| z \in \mathbb{R}^K, 0 < z\right\}$. $Y, \overline{Y}$ are in fact convex and obviously disjoint. The hyperplane which separates Y and $\overline{Y}$ generates a linear functional with the desired properties.

The following corollary is an easy consequence of proposition 8.

Corollary 5: If $M(X) \subset \mathbb{R}^K$ is a subspace, that is all (actively traded) portfolios may be sold short, then the pricing functional ϕ is a **linear** functional.

[59] For their no–arbitrage definition Jouini/Kallal (1995a), p. 205 derive a similar result.

[60] See Franklin (1980), p. 49, theorem 3.

References

Black, Fischer/Scholes, Myron (1973), The Pricing of Options and Corporate Liabilities, in: Journal of Political Economy, Vol. 81, pp. 637–654.

Cox, John C./Ross, Stephen A./Rubinstein, Mark (1979), Option Pricing: A Simplified Approach, in: Journal of Financial Economics, Vol. 7, pp. 229–263.

Delbaen, Freddy (1992), Representing Martingale Measures when Asset Prices are Continuous and Bounded, in: Mathematical Finance, Vol. 2, pp. 107–130.

Delbaen, Freddy/Schachermayer, Walter (1994), A General Version of the Fundamental Theorem of Asset Pricing, in: Mathematische Annalen, Vol. 300, pp. 463–520.

Detemple, Jerome/Murthy, Shashidhar (1997), Equilibrium Asset Prices and No–Arbitrage with Portfolio Constraints, in: The Review of Financial Studies, Vol. 10, pp. 1133–1174.

Duffie, Darrell (1996), Dynamic Asset Pricing Theory, 2nd edition.

Dybvig Philip H./Ingersoll, Jonathan E. Jr. (1982), Mean–Variance Theory in Complete Markets, in: The Journal of Business, Vol. 55, pp. 233–251.

Dybvig, Philip H./Ross, Stephen A. (1992), Arbitrage, in: Palgrave, R. H./ Newman, P. (eds.), The New Palgrave Dictionary of Money & Finance A – I, pp. 43–50.

Franklin, Joel (1980), Methods of Mathematical Economics.

Garman, Mark B./Ohlson, James A. (1981), Valuation of Risky Assets in Arbitrage-free Economies with Transactions Costs, in: Journal of Financial Economics, Vol. 9, pp. 271–280.

Harrison, J. Michael/Kreps, David M. (1979), Martingales and Arbitrage in Multiperiod Securities Markets, in: Journal of Economic Theory, Vol. 20, pp. 381–408.

Harrison, J. Michael/Pliska, Stanley R. (1981), , Martingales and Stochastic Integrals in the Theory of Continuous Trading, in: Stochastic Processes and their Applications, Vol. 11, pp. 215–260.

Harrison, J. Michael/Pliska, Stanley R. (1983), A Stochastic Calculus Model of Continuous Trading: Complete Markets, in: Stochastic Processes and their Applications, Vol. 15, pp. 313–316.

Hull, John C. (1997), Options, Futures, and Other Derivatives, 3rd edition.

Ingersoll, Jonathan E., Jr. (1987), Theory of Financial Decision Making.

Jevons, William S. (1871), The Theory of Political Economy.

Jouini, Elyés/Kallal, Hédi (1995b), Arbitrage in Securities Markets with Short–Sales Constraints, in: Mathematical Finance, Vol. 5, pp. 197–232.

Jouini, Elyés/Kallal, Hédi (1995b), Martingales and Arbitrage in Securities Markets with Transaction Costs, in: Journal of Economic Theory, Vol. 66, pp. 178–197.

Koehl, Pierre-F./Pham, Huyên (2000), Sublinear Price Functionals under Portfolio Constraints, in: Journal of Mathematical Economics, Vol. 33, pp. 339–351.

Kreps, David M. (1981), Arbitrage and Equilibrium in Economies with Indefinitely Many Commodities, in: Journal of Mathematical Economics, Vol. 8, pp. 15–35.

Merton, Robert C. (1973), Theory of Rational Option Pricing, in: Bell Journal of Economics and Management Science, Vol. 4, pp. 141–183.

Munn, Glenn G. (1983), Encyclopedia of Banking and Finance, 8th edition.

Musiela, Marek/Rutkowski, Marek (1997), Martingale Methods in Financial Modelling.

Oh, Gyutaeg/Ross, Stephen A./Shanken, Jay (1998), The Absence of Arbitrage: Some New Results, Working Paper (current revision: September 1998), University of Rochester.

Ross, Stephen A. (1978), A Simple Approach to the Valuation of Risky Streams, in: The Journal of Business, Vol. 51, pp. 453–475.

Schäfer, Dorothea/Kruschwitz, Lutz/Schwake, Mike (1998), Studienbuch Finanzierung und Investition, 2nd edition.

Schöbel, Rainer (1995), Kapitalmarkt und zeitkontinuierliche Bewertung.

Wilhelm, Jochen (1981), Zum Verhältnis von Capital Asset Pricing Model, Arbitrage Pricing Theory und Bedingungen der Arbitrage–Freiheit von Finanzmärkten, in: Schmalenbachs Zeitschrift für betriebswirtschaftliche Forschung, Vol. 33, pp. 891–905.

Wilhelm, Jochen (1983), Finanztitel und Unternehmensfinanzierung.

Wilhelm, Jochen (1987), Arbitrageorientierte Bewertung von Finanztiteln: Eine Neuformulierung, in: Heilmann, W.–R. et al. (eds.), Geld, Banken und Versicherungen, Vol. 2, pp. 1155–1177.