AN OPTION-PRICING FRAMEWORK FOR THE VALUATION OF FUND MANAGEMENT COMPENSATION

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ABSTRACT

Compensation of funds managers increasingly involves elements of profit sharing that entitle managers to option-like payoffs. An important example is the compensation of private equity fund managers. Compensation of private equity fund managers typically consists of a fixed management fee and a performance-related carried interest. The fixed management fee resembles common compensation terms of mutual funds and hedge funds, while the performance-related carried interest is uncommon among most mutual funds. Moreover, the performance-related carried interest typically differs from variable hedge fund fees. In this chapter, we derive the value of the variable components of private equity fund managers’ compensation based on a risk-neutral option-pricing approach.

1. INTRODUCTION

Compensation of funds managers increasingly involves elements of profit sharing that entitle managers to option-like payoffs that are contingent on the performance of the managed fund. An important example is the compensation of private equity fund managers, to which only limited research has been devoted yet. In this chapter, we show how to value the variable components of private equity fund managers’ compensation based on a risk-neutral option-pricing approach.
Private equity funds are typically organized as limited partnerships, with private equity firms serving as General Partners (GPs) of the funds, and large institutional investors and wealthy individuals providing bulk of the capital known as Limited Partners (LPs). These limited partnerships typically last for 10 years, and partnership agreements signed at a fund initiation define the expected payments to GPs. These payments consist of two components. A fixed component (called “management fee”) and a performance related component (called “carried interest” or simply “carry”). The fixed management fee resembles well pricing terms of mutual funds and hedge funds, while the performance related carried interest is uncommon among most mutual funds and is different from the variable incentive fees of hedge funds.

For the GPs, carried interest is an option-like position on the total proceeds of all investments of the fund. In practice, multiple, staggered investments and stepwise liquidation of the investments make carried interest considerably more complex than a simple call option on a traded asset. In particular, the option payoffs are typically contingent upon the performance of the fund, where performance is typically measured by the internal rate of return of the fund cash flows. To derive solutions to the problem of valuing these option-like positions, we first develop a continuous-time model of the cash flow dynamics of private equity funds. This model consists of three main components. (1) The modeling of the capital drawdowns, (2) the modeling of the investment value and (3) the modeling of the return repayments. To keep the model simple, a geometric Brownian motion is utilized for the dynamics of the investment value and two mean-reverting square root processes are applied for the dynamics of the drawdown and the distribution dynamics. Second, we use risk-neutral valuation to estimate the value of the carried-interest option at the fund inception date. This is conducted using a numerical Monte-Carlo simulation of the fund cash flows.

This chapter is related to the literature that investigates the fee terms of private equity partnerships. The first paper to address this issue is Gompers and Lerner (1999a), who explore cross-sectional and time-series variation in the terms of venture capital funds. Similar issues are addressed by Litvak (2004), who extends the analysis of Gompers and Lerner (1999a) from a more legal perspective. Neither of these papers uses an option-pricing framework to value the carried interest. An exception to this is Metrick and Yasuda (2010), who also use a risk-neutral valuation framework to estimate the value of various pricing terms. However, they do not develop a comprehensive model for the cash flow and value dynamics of private equity funds, as done in this chapter. As shown, this framework allows us to analyze how the value of the pricing terms is related to the cash flow dynamics and the risk/return characteristics of a fund. This gives
rise to several important determinants of fund fee value that have not previously been studied.

The remainder of this chapter is organized as follows. In the next section, we set forth the notation, assumptions, and structure of the model underlying our fee valuation. Section 3 presents the risk-neutral valuation framework for lifetime private equity fund fees. Section 4 presents the results of a model calibration and analyzes determinants of private equity fund fee value. The conclusions are presented in Section 5.

2. THE FRAMEWORK

The aim of this section is to develop the framework in which we derive the value of private equity fund fees. We start with a brief description that lays out the typical construction of private equity funds and explain the two components of private equity fund fees. This gives the motivation for the subsequent model of the dynamics of private equity funds.

2.1 Institutional Setting

Investments in private equity are typically intermediated through private equity funds. Thereby, a private equity fund denotes a pooled investment vehicle whose purpose is to negotiate purchases of common and preferred stocks, subordinated debt, convertible securities, warrants, futures and other securities of companies that are usually unlisted. As the vast majority of private equity funds, the fund to be modeled here is organized as a limited partnership in which the private equity firm serves as the general partner (GP). The bulk of the capital invested in private equity funds is typically provided by institutional investors, such as endowments, pension funds, insurance companies, and banks. These investors, called limited partners (LPs), commit to provide a certain amount of capital to the private equity fund (the so-called committed capital). The GP then has an agreed time period in which to invest this committed capital – usually on the order of five years. This time period is commonly referred to as the fund commitment period. In general, when a GP identifies an investment opportunity, he “calls” money from the LPs up to the amount committed, and a GP can do so at any time during the pre-specified commitment period. The capital calls usually occur unscheduled over the commitment period, where the exact timing only depends on the investment decisions of the GPs. However, total capital calls over the commitment period do not exceed the total committed capital. The capital calls are also called drawdowns or takedowns. As those drawdowns occur, the available cash is
immediately invested in managed assets and the size of the portfolio begins to increase. When an investment is liquidated, the GP distributes the proceeds to its LPs either in marketable securities or in cash. The GP also has an agreed time period in which to return capital to the LPs – usually on the order of ten to fourteen years. This time period is also called the total legal lifetime of the fund.\(^1\)

Following the typical structure of private equity funds, the GPs receive two types of compensations for managing the investments: a fixed component (called “management fee”) and a performance related component (called “carried interest” or simply “carry”).\(^2\)

The management fee is generally expressed as a percentage of the committed capital and is paid annually. Typically, the fees are around 2 percent yearly and vary based on fund sizes (see Metrick and Yasuda (2010)). Often the fees decrease after the commitment period to reflect the fact that less time is required in managing activities, especially when investments are mature or partly realized. Tapering the management fee is effected by either reducing the accounting basis on which the rate is applied -or through reduction of the percentage rate paid for compensation. Tapered fee corresponds to the real purpose of the management fee to cover the costs of running and administering the fund. For example, if a fund charges a fixed 2 percent annual management fees on committed capital for the entire fund lifetime of 10 years, then the lifetime management fees of the fund amount to 20 percent of the committed capital. Thus only the remaining 80 percent of the committed capital will be available for investments into portfolio companies. We will denote this fraction as fund investment capital in the following. Using these definitions, total fund committed capital can be split into two components according to:

\[
\text{Committed Capital} = \text{Investment Capital} + \text{Lifetime Management Fees.}
\]

The second source of compensation is the carried interest or carry, which entitles venture capital firms a certain share of the capital gains from the fund. Typically, the carried interest is only paid if LPs received initial investment plus some other form of pre-specified interests. Most LPs require interests on their capital known as a hurdle rate. This hurdle rate is not a guaranteed interest payment to the fund providers, but rather represents target returns above which venture capital firms are entitled to receive the carried interest. The hurdle rate is typically 8 percent and the carried interest level as prevailed in the industry is 20 percent (see Metrick and Yasuda (2010)).
Table 1 illustrates the computation of the carried interest using a simple numerical example. Assume a private equity fund with a committed capital of $100M, a carried interest level of 20 percent, a hurdle rate of 8 percent, and a fixed lifetime of ten years. For simplicity, we assume that there are no management fees and that the committed capital of the fund is fully drawn by the fund and hence, the absolute value of the negative cash flows during the first four years represents committed capital (i.e. $100M). The fund has exited some of their portfolio companies beginning of year five and continued to the end of year 10. The table shows that the fund generates profits of $150M net of investments or equivalently a multiple of 2.5 (250/100) at the end of fund life.

The calculated cash flow based internal rates of return (IRR) show that the distribution schedule of the fund satisfies the 8 percent hurdle rate requirement in year seven. One year later, the GPs are entitled to receive carried interest. The exact amount of the carried interest is thereby affected by the existence of a catch-up clause, as shown in Table 1. With no catch-up clause, the GPs receive 20 percent of the capital distributions beginning of year eight and continued to the end of year 10. This results in total carried interest payments of $22M, which is lower than the carried interest level of 20 percent times the net profits of $150M of the fund. Thus the GPs receive a lower fraction of the net profits of the fund than the defined carried interest level when no catch-up clause exists. The key idea of a catch-up clause is that the GPs of a fund with a catch-up and hurdle rate receive a fraction of the profits of the fund equal to the carried interest level as long as the fund is sufficiently profitable (i.e., the fund has an IRR that exceeds the hurdle rate). Table 1 shows that the catch-up provision in this example implies that the GPs receive $18M carried interest in year eight, which is equal to the carried interest level of 20 percent times the net profits of the fund of $90M ($190M of distributions on $100M of committed capital) up to year eight. After year eight, the GPs also receive a fraction of 20 percent of the capital
distributions. This results in total carried interest payments of $30M, which is exactly the carried interest level of 20 percent times the net profits of $150M of the fund. In this example, we have chosen a catch-up provision of 100 percent. A fund with a catch-up percentage below 100 percent will still receive a proportion equal to the carried interest level of the fund profits, but at a somewhat slower pace than in the example shown here.

The above description has made clear that carried interest is an option-like position on the total proceeds of the fund investments. Valuing such a contingent claim requires modeling the cash flow dynamics of private equity funds.

2.2 Private Equity Fund Dynamics

Following the typical construction of private equity funds outlined above, modeling the cash flow dynamics of private equity funds requires three main ingredients: the modeling of the capital drawdowns, the modeling of the investment value and the modeling of the return repayments.

A. Capital Drawdowns

We consider a private equity fund with a total legal lifetime of \( T_f \). We begin by assuming that the fund to be modeled has total commitments given by \( CO \), total lifetime management fees given by \( MF_{T_f} \), and an investment capital given by \( IC \). As defined above, it must hold that

\[
CO = IC + MF_{T_f},
\]

(1)

Cumulated capital drawdowns from the LPs up to some time \( t \) during the commitment period \( T_c \) are denoted by \( D_t \), undrawn capital up to time \( t \) by \( U_t \). When the fund is set up at \( t = 0 \), \( D_0 = 0 \) and \( U_0 = IC \) are given by definition. Furthermore, at any time \( t \in [0, T_c] \), the simple identity

\[
D_t = IC - U_t
\]

(2)

must hold. In the following, assume capital to be drawn over time at some non-negative rate from the remaining undrawn investment capital \( U_t = IC - D_t \).

**Assumption 2.1** The dynamics of the cumulated capital drawdowns \( D_t \) can be described by the ordinary differential equation

\[
dD_t = \delta_t U_t 1_{[0\le t \le T_c]} dt,
\]

(3)

where \( \delta_t \ge 0 \) denotes the rate of contribution, or simply the fund’s drawdown rate.
In most cases, capital drawdowns of private equity funds are heavily concentrated in the first few years or even quarters of a fund’s life. After high initial investment activity, drawdowns of private equity funds are carried out at a declining rate, as fewer new investments are made, and follow-on investments are spread out over a number of years. This typical time-pattern of the capital drawdowns is well reflected in the structure of equation (3). Under the specification (3), cumulated capital drawdowns \( D_t \) are given by

\[
D_t = IC - IC \exp \left( - \int_0^{t \leq T_c} \delta_u \, du \right). \tag{4}
\]

Equation (4) shows that the initially high capital drawdowns at the start of the fund decrease over the commitment period \( T_c \) of the fund. This follows as undrawn amounts of the investment capital, \( U_t = IC \exp \left( - \int_0^{t \leq T_c} \delta_u \, du \right) \), decay exponentially over time. A condition that leads to the realistic feature that capital drawdowns are highly concentrated in the early times of a fund’s life. Furthermore, equation (4) shows that the cumulated drawdowns \( D_t \) can never exceed the total investment capital \( IC \) under this model setup, i.e., \( D_t \leq IC \) holds for all \( t \in [0, T_c] \). At the same time the model also allows for a certain fraction of the investment capital \( IC \) not to be drawn, as the commitment period \( T_c \) represents a cut-off point for capital drawdowns.

As investment opportunities typically do not arise constantly over the commitment period \( T_c \), we introduce a stochastic process for the drawdown rate \( \delta_t \).

**Assumption 2.2** The drawdown rate \( \delta_t \) is given by a stochastic process \( \{ \delta_t, 0 \leq t \leq T_c \} \). Its specification under the objective probability measure \( \mathbb{P} \) is

\[
d\delta_t = \kappa_1 (\theta_1 - \delta_t) \, dt + \sigma_1 \sqrt{\delta_t} \, dB_{1,t}, \tag{5}
\]

where \( \theta_1 > 0 \) is the long-run mean of the drawdown rate, \( \kappa_1 > 0 \) governs the rate of reversion to this mean, \( \sigma_1 > 0 \) reflects the volatility of the drawdown rate, and \( B_{1,t} \) is a standard Brownian motion.

The drawdown rate behavior implied by the above square-root diffusion ensures that negative values of the drawdown rate are precluded\(^3\) and that the drawdown rate randomly fluctuates around some mean level \( \theta_1 \). We are now equipped with the first component of our model. The following turns to the modeling of the capital distributions.

**B. Capital Distributions**

As capital drawdowns occur, the available capital is immediately invested in managed assets and the portfolio of the fund begins to accumulate. As the
underlying investments of the fund are gradually exited, cash or marketable securities are received and finally returns and proceeds are distributed to the LPs. Let cumulated capital distributions up to some time \( t \in [0, T_i] \) during the legal lifetime \( T_i \) of the fund be denoted by \( R_t \). Recognizing that the size and timing of repayments are based on the performance of the fund, it is assumed that capital distributions occur at a non-negative rate \( \rho_t \) from the total investment portfolio value \( V_t \) of the fund at time \( t \).

**Assumption 2.3** The dynamics of the cumulated capital distributions \( R_t \) can be described by

\[
dR_t = \rho_t V_t \, dt, \quad \text{if} \quad t < T_i, \quad \text{and} \quad R_t = \int_0^t \rho_u V_u \, du + V_t 1_{\{t = T_i\}},
\]

where \( \rho_t \geq 0 \) denotes the rate of repayment, or simply the fund’s distribution rate.

The ordinary differential equation in specification (6) illustrates that capital repayments occur at a non-negative rate \( \rho_t \). This, however, holds only in case \( t < T_i \). As funds are fully liquidated at the end of their legal lifetime, cumulated capital distributions over the entire life of the fund must also include the final reimbursement of the assets of the fund at maturity \( T_i \), i.e., \( R_{T_i} = \int_0^{T_i} \rho_u V_u \, du + V_{T_i} \). Note also that the model developed here allows cumulated distributions to be greater (or smaller) than the cumulated drawdowns, implying a possibly positive (or negative) performance of the fund over its lifetime.

To capture the erratic feature of real world private equity fund capital distribution, we also introduce a stochastic process for the distributions rate \( \rho_t \).

**Assumption 2.4** The distribution rate \( \rho_t \) is given by a stochastic process \( \{\rho_t, 0 \leq t \leq T_i\} \). Its specification under the objective probability measure \( \mathbb{P} \) is

\[
d\rho_t = \kappa_2 (\theta_2 - \rho_t) \, dt + \sigma_2 \sqrt{\rho_t} \, dB_{2,t},
\]

where \( \theta_2 > 0 \) is the long-run mean of the distribution rate, \( \kappa_2 > 0 \) governs the rate of reversion to this mean, \( \sigma_2 > 0 \) reflects the volatility of the distribution rate, and \( B_{2,t} \) is a second standard Brownian motion.

We model distributions and drawdowns separately and, therefore, must similarly restrict capital distributions to be strictly non-negative at any time \( t \) during the legal lifetime \( T_i \) of the fund. This is again achieved by assuming a square-root diffusion for the dynamics of the distributions rate. Additionally, the specification (7) has the attractive feature that the distributions rate will randomly fluctuates around the mean level \( \theta_2 \).
Besides the stochastic distributions rate, \( \rho_t \), capital distributions of a fund are also driven by the evolution of the fund value \( V_t \) over time, which is specified in the following.

C. Investment Value
The last step in the modeling of private equity funds is characterizing the dynamics of the value of the fund \( V_t \) over the legal lifetime \( T_t \). Fund values are affected by three variables: capital drawdowns, capital distributions and investment performance. It is assumed that the return on any cash flow invested in the fund can be described by a normal distribution with constant mean \( \mu \) and constant (non-negative) volatility \( \sigma \).

**Assumption 2.5** The dynamics of the fund values \( V_t \) under the objective probability measure \( \mathbb{P} \) can be described by the stochastic differential equation

\[
dV_t = \mu V_t dt + \sigma V_t dB_{3,t} + dD_t - dR_t,
\]

where \( \mu \) is the constant mean rate of return, \( \sigma > 0 \) is the constant return volatility, and \( B_{3,t} \) is a third standard Brownian motion.

From specification (8), one can infer that the change in value of the fund is made of the performance of the existing investment \( V_t \). Randomness in returns is introduced into the model by the standard Brownian motion \( B_t \). Besides, the value of the fund is augmented by the capital drawdowns and decreased by the capital distributions.

Substituting the definition of \( dR_t \), equation (8) can be simplified to

\[
dV_t = (\mu - \rho_t)V_t dt + \sigma V_t dB_{3,t} + dD_t, \quad \text{if} \quad t < T_t,
\]

which is the stochastic differential equation of a so-called inverse gamma process that turns into a standard geometric Brownian motion in case of \( t > T_t \). Note here that final value is \( V_{T_t} = 0 \) because of the final cash flow repayment inherent in specification (6). In addition, note that also \( V_0 = 0 \) holds in the model by definition.

3. **RISK-NEUTRAL FEE VALUATION**

In this section, we derive the value of lifetime private equity fund fees using risk-neutral pricing techniques. We start with a formal definition of the two fee components. Then, the risk-neutral valuation framework is presented.
3.1 Definition of Fee Components

As outlined in Section 2.1, GPs typically receive two forms of compensations for managing the partnership interests. The first, is fixed revenue and paid as management fees, while the second is performance-based revenue and payable as carried interest.

The formal definition of the management fees in a continuous-time framework is straightforward. Let $MF_t$ denote cumulated management fees up to some time $t \in [0,T_t]$. If management fees are defined as a percentage $c_{mf}$ of the committed capital $CO$ and paid continuously, the dynamics of the management fees can be represented by the ordinary differential equation

$$dMF_t = c_{mf}CO dt.$$  \hspace{1cm} (10)

The definition of the carried interest is slightly more complex as this performance based compensation depends on the fund cash flows. Let $CI_t$ denote the cumulated carried interest up to some time $t \in [0,T_t]$. The carried interest entitles the GPs a certain share of the capital gains from the fund. If the carried interest level is given by $c_{ci}$, the GPs receive the fraction $c_{ci}$ of the net cash flows of the fund after management fees. Mathematically, the dynamics of the carried interest can be described by

$$dCI_t = c_{ci}\max\{dR_t - dD_t - dMF_t, 0\}1_{(IRR_t > h)},$$ \hspace{1cm} (11)

where taking the maximum of the net cash flows after management fees and zero, $\max\{dR_t - dD_t - dMF_t, 0\}$, ensures that instantaneous carried interest payments are non-negative. In addition, multiplying by the indicator function $1_{(IRR_t > h)}$ guarantees that the performance related carried interest is only payable at time $t$ if the internal rate of return of the fund at that time, $IRR_t$, exceeds the specified hurdle rate $h$, i.e. $IRR_t > h$ holds. The internal rate of return is the performance measure commonly employed in the private equity industry to evaluate the return of a fund. In continuous-time, the internal rate of return $IRR_t$ of the net fund cash flows after management fees at time $t$ is a solution to

$$\int_0^t e^{-IRR_t u}(dR_u - dD_u - dMF_u) = 0.$$ \hspace{1cm} (12)

Equation (11) implicitly assumes that the carried interest is paid without the existence of a catch-up clause. If the carried interest is paid with a 100 percent catch-up provision, the dynamics of the carried interest become more complex. Again, carried interest is only payable at time $t$ if $IRR_t > h$ holds. In addition, if the fraction of the cumulated carried interest of the net profits of the fund is equal to the carried interest level, i.e. $CI_t / (R_t - CO) = c_{ci}$ holds, the GPs receive an instantaneous carried interest of $c_{ci}\max\{dR_t - dD_t - dMF_t, 0\}$. This is similar to the definition of the carried interest without a catch-up in equation (11). In
contrast, if \( C_{t}/(R_{t} - CO) < c_{ci} \) holds, instantaneous carried interest payments are equal to \( \min\{c_{ci}(R_{t} - CO) - CI_{t}, dR_{t} - dD_{t} - dMF_{t}\} \). Thereby, taking the minimum here means that the catch-up payments, \( c_{ci}(R_{t} - CO) - CI_{t} \), cannot exceed instantaneous net fund cash flows after management fees, \( dR_{t} - dD_{t} - dMF_{t} \).

3.2 Risk-Neutral Valuation

Based on our stochastic cash flow models and the definitions made above, we can now derive the value of lifetime fund fees of the GPs. The value of the outstanding fund fees \( V_{t}^{GP} \) at time \( t \in [0, T_{t}] \) is defined by the discounted value of all expected outstanding lifetime fund fees, including management fees and the carried interest. Applying a risk-neutral valuation approach, the arbitrage-free value of the fund fees is given by

\[
V_{t}^{GP} = V_{t}^{MF} + V_{t}^{CI} = \mathbb{E}^{Q}_{t} \left[ \int_{t}^{T_{t}} e^{-r_{f}(u-t)} dMF_{u} \right] + \mathbb{E}^{Q}_{t} \left[ \int_{t}^{T_{t}} e^{-r_{f}(u-t)} dCI_{t} \right]. \tag{13}
\]

where \( \mathbb{E}^{Q} \) denotes the expectations operator conditional on the information set available at time \( t \). This conditional expectations operator is defined under the risk-neutral probability measure \( Q \). Therefore, discounting at the riskless rate \( r_f \) is appropriate in equation (13). From specification (13), the value of lifetime fund fees consists of two components. The first term, \( V_{t}^{MF} \), is the value of the outstanding management fees. The second term, \( V_{t}^{CI} \), corresponds to the value of the outstanding carried interest.

The evaluation of the first integral for the value of the management fees is trivial. This is since the fixed management fee is a contractual arrangement that is paid riskless and by a constant amount per time period. Therefore, we can eliminate the first expectation in (13), as the expectation of a constant is simply the constant itself. Substituting equation (10), the value of the outstanding management fees turns out to be

\[
V_{t}^{MF} = c_{mf} CO \left[ \int_{t}^{T_{t}} e^{-r_{f}(u-t)} du \right] = c_{mf} CO \frac{1-e^{-r_{f}(T_{t}-t)}}{r_{f}}. \tag{14}
\]

Substituting equation (11), the value of the outstanding carried interest (with no catch-up) can be evaluated by solving

\[
V_{t}^{CI} = \mathbb{E}^{Q}_{t} \left[ \int_{t}^{T_{t}} e^{-r_{f}(u-t)} c_{ci} \max\{dR_{u} - dD_{u} - dMF_{u}, 0\} 1_{(IRR_{u} > h)} \right]. \tag{15}
\]

From equation (15), one can directly infer that the value of the carried interest is a contingent claim on the capital drawdowns and distributions of a private equity fund. The complication here arises from the fact that the state variables underlying the valuation, i.e. the assumed cash flow processes, do not represent traded assets. In such an incomplete market setting, risk-neutral pricing based on
arbitrage considerations alone is not feasible. Hence, the risk sources underlying our model have to be transformed.

Applying Girsanov's Theorem, as for example outlined in Duffie (2001), it follows that the underlying stochastic processes for the drawdown rate, distribution rate, and the investment value under the risk-neutral probability measure \( \mathbb{Q} \) are given by

\[
d\delta_t = \left[ \kappa_1 (\theta_1 - \delta_t) - \lambda_1 \sigma_1 \sqrt{\delta_t} \right] dt + \sigma_1 \sqrt{\delta_t} \, dB_{1,t}^\mathbb{Q},
\]

\[
d\rho_t = \left[ \kappa_2 (\theta_2 - \rho_t) - \lambda_2 \sigma_2 \sqrt{\rho_t} \right] dt + \sigma_2 \sqrt{\rho_t} \, dB_{2,t}^\mathbb{Q},
\]

\[
dV_t = (\mu - \lambda_3 \sigma) V_t dt + \sigma V_t dB_{3,t}^\mathbb{Q} - dR_t + dD_t,
\]

where \( B_{1,t}^\mathbb{Q}, B_{2,t}^\mathbb{Q} \) and \( B_{3,t}^\mathbb{Q} \) are \( \mathbb{Q} \)-Brownian motions and \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are the corresponding market prices of risk. To simplify the valuation, we assume that the drawdown and distribution rate carry zero systematic risk. Under this condition, the market prices of risk \( \lambda_1 \) and \( \lambda_2 \) are equal to zero and the drift coefficients of the stochastic processes \( \delta_t \) and \( \rho_t \) are not affected by the change of the probability measure.

The third market price of risk \( \lambda_3 \) can be derived using a version of the intertemporal capital asset pricing model (ICAPM) of Merton (1973). It follows that the market price of risk is given by:

\[
\lambda_3 = \text{Corr}(dV_t, dM_t) \text{Std}(dM_t).
\]

That is, the market price of risk is simply computed by the correlation \( \text{Corr}(dV_t, dM_t) \) between the fund returns and the market portfolio returns multiplied by the volatility \( \text{Std}(dM_t) \) of the market portfolio returns.

Unfortunately, valuation equation (15) cannot be solved analytically because of the path-dependent structure of the carried interest that is contingent on the internal rate of return of the fund. Therefore, we have to resort to a numerical technique. This is done here by using Monte Carlo simulations based on discrete-time approximations of the stochastic cash flow and fee model. The details of the numerical procedure to solve equations (15) are outlined in the Appendix.

4. NUMERICAL EXAMPLE

In this section, the fee valuation is applied using a set of reasonable input parameters. We assume a fund with a committed capital equal to $100 and a legal lifetime of 10 years. The fee terms are a 2 percent management fee, a carried interest level of 20 percent, and a hurdle rate of 8 percent. For the dynamics of the capital drawdowns and capital distribution, we rely on the parameters estimated by Malherbe (2004) for the buyout segment: \( \kappa_1 = 8.74, \theta_1 = 0.32, \sigma_1 = 1.46 \),
NUMERICAL EXAMPLE

For the value dynamics, we assume an expected return of \( \mu = 0.15 \) and a volatility of \( \sigma = 0.30 \). To compute the market price of risk \( \lambda_3 \), we assume a correlation coefficient of 0.5 between the fund and market returns and a volatility of the market returns equal to 0.2. For this set of model parameters, the market price of risk turns out to be \( \lambda_3 = 0.1 \). Finally, the riskless rate of interest is set to 5 percent.

Table 2 summarizes the fee values that can be computed using this set of model parameters. All fee values are expressed in dollars per $100 of committed capital. The table shows that the total value of lifetime fees is $19.66 for a fund with no catch-up and $20.02 for a fund with a 100 percent catch-up provision. That is, total value of lifetime fee revenues amounts to around 20 percent of the committed capital of the fund. Management fees account for the largest portion of the value of the lifetime fees and are not affected by the existence of a catch-up clause. In both categories, close to four-fifth of the value of the total revenues derive from the fixed management fee and only one-fifth from the variable revenue generated by the carried interest payments. As expected, the existence of a catch-up clause increases the value of the carried interest. However, the figures provided in Table 2 imply that the magnitude of this effect is relatively low.

Besides calculating fee values it is also interesting to explore how they are affected by the various model parameters. The factors influencing the value of the outstanding management fees, \( V_t^{MF} \), can directly be inferred from equation (15). Table 3 summarizes these factors and indicates their directional influence on the value of the outstanding management fees. As shown in the table, the value increases as the remaining fund lifetime \( T_f - t \), the committed capital \( C \), or the management fee level \( c_{mf} \) are increased. In contrast, the value of outstanding management fees decreases with higher levels of the riskless rate \( r_f \), as this leads to higher discounting of the outstanding management fees.

### Table 2. Estimated Fee Values

This table summarizes the outputs of the fee valuation. Fee values are expressed in dollars per $100 of committed capital. The fee terms employed for the calculations are a 2 percent management fee, a carried interest level of 20 percent, and a hurdle rate of 8 percent. Calculations are shown for a fund with no catch-up clause and fund with a catch-up clause of 100 percent.

<table>
<thead>
<tr>
<th>Management Fee Value</th>
<th>Carried Interest Value</th>
<th>Total Fee Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Catch-Up</td>
<td>15.86</td>
<td>3.80</td>
</tr>
<tr>
<td>With Catch-Up</td>
<td>15.86</td>
<td>4.16</td>
</tr>
</tbody>
</table>

\( \kappa_2 = 17.47, \theta_2 = 0.20, \) and \( \sigma_2 = 1.93 \). For the value dynamics, we assume an expected return of \( \mu = 0.15 \) and a volatility of \( \sigma = 0.30 \). To compute the market price of risk \( \lambda_3 \), we assume a correlation coefficient of 0.5 between the fund and market returns and a volatility of the market returns equal to 0.2. For this set of model parameters, the market price of risk turns out to be \( \lambda_3 = 0.1 \). Finally, the riskless rate of interest is set to 5 percent.

Table 2 summarizes the fee values that can be computed using this set of model parameters. All fee values are expressed in dollars per $100 of committed capital. The table shows that the total value of lifetime fees is $19.66 for a fund with no catch-up and $20.02 for a fund with a 100 percent catch-up provision. That is, total value of lifetime fee revenues amounts to around 20 percent of the committed capital of the fund. Management fees account for the largest portion of the value of the lifetime fees and are not affected by the existence of a catch-up clause. In both categories, close to four-fifth of the value of the total revenues derive from the fixed management fee and only one-fifth from the variable revenue generated by the carried interest payments. As expected, the existence of a catch-up clause increases the value of the carried interest. However, the figures provided in Table 2 imply that the magnitude of this effect is relatively low.

Besides calculating fee values it is also interesting to explore how they are affected by the various model parameters. The factors influencing the value of the outstanding management fees, \( V_t^{MF} \), can directly be inferred from equation (15). Table 3 summarizes these factors and indicates their directional influence on the value of the outstanding management fees. As shown in the table, the value increases as the remaining fund lifetime \( T_f - t \), the committed capital \( C \), or the management fee level \( c_{mf} \) are increased. In contrast, the value of outstanding management fees decreases with higher levels of the riskless rate \( r_f \), as this leads to higher discounting of the outstanding management fees.
Table 4 summarizes the parameters influencing the value of the carried interest, $V_t^{ci}$. First, the value of the carried interest depends on the fund terms. Similar to the value of the management fees, the value of the carried interest increases as the remaining fund lifetime $T_t - t$ or the committed capital $C$ increases. The carried interest entitles the GPs a percentage of the capital gains of the fund equal to the carried interest level $A$ if the fund’s return exceeds the hurdle rate $h$. Therefore, the value increases (decreases) with an increasing carried interest level $A$ (hurdle rate $h$).

Second, the value of the carried interest is also influenced by the parameters governing the stochastic dynamics of the drawdown and distribution rate. The main model parameters affecting the carried interest value here are the long-term drawdown rate $\theta_1$ and the long-term distribution rate $\theta_2$. A higher long-term drawdown rate $\theta_1$ increases the value of the carried interest. This holds as a higher parameter $\theta_1$ on average leads to a faster drawdown schedule. A faster drawdown schedule in turn results in earlier carried interest payments on average. Through the effect of a lower discounting of earlier carried interest payments this results in higher carried interest values. Conversely, increasing the long-term distribution rate $\theta_2$ decreases the value of the carried interest. This holds because a higher level of $\theta_2$ decreases the average period over which the capital is tied up in portfolio companies which also decreases the probability of fund returns that exceed the hurdle rate. Increasing the volatility parameters $\sigma_1$ and $\sigma_2$ has a similar effect on the value of the carried interest. However, note that the impact of a variation of $\sigma_1$ or $\sigma_2$ is much smaller than the impact of a variation of $\theta_1$ or $\theta_2$. The impact is about the same relative magnitude than that of a variation of the speed of adjustment parameters $\kappa_1$ and $\kappa_2$. The speed of adjustment parameterst have an opposite effect on the carried interest value which can be explained by the fact that a high speed of adjustment can absorb some of the volatility of the drawdown and distribution rate.

### Table 3. Factors Determining Management Fee Value

This table summarizes the factors determining the value of the outstanding management fees. The column labeled direction indicates how the value of the management fees changes if the corresponding parameter is increased.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Remaining fund lifetime</td>
<td>$T_t - t$</td>
<td>Increases</td>
</tr>
<tr>
<td>Committed capital</td>
<td>$C$</td>
<td>Increases</td>
</tr>
<tr>
<td>Management fee level</td>
<td>$c_{mf}$</td>
<td>Increases</td>
</tr>
<tr>
<td>Riskless rate</td>
<td>$r_f$</td>
<td>Decreases</td>
</tr>
</tbody>
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Table 4 summarizes the parameters influencing the value of the carried interest, $V_t^{ci}$. First, the value of the carried interest depends on the fund terms. Similar to the value of the management fees, the value of the carried interest increases as the remaining fund lifetime $T_t - t$ or the committed capital $C$ increases. The carried interest entitles the GPs a percentage of the capital gains of the fund equal to the carried interest level $A$ if the fund’s return exceeds the hurdle rate $h$. Therefore, the value increases (decreases) with an increasing carried interest level $A$ (hurdle rate $h$).

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Third, the value of the carried interest is influenced by the asset value dynamics and the riskless rate of interest. The higher the expected return of the assets $\mu$, the more likely the fund’s return will exceed the defined hurdle rate, $h$. Therefore, a higher expected rate of return $\mu$ increases the value of the carried interest. Likewise, a higher asset volatility $\sigma$ increases the probability of high carried interest payments and thus increases the value of the carried interest. Finally, the value of the carried interest decreases with higher levels of the riskless rate $r_f$, as this parameter leads to higher discounting of future carried interest payments.

### Table 4. Factors Determining Carried Interest Value

This table summarizes the main factors determining the value of the outstanding carried interest. The column labeled direction indicates how the value of the carried interest changes if the corresponding parameter is increased. For parameters with only a small impact on the carried interest value, the direction is shown in brackets. Note that the relationships shown in this table hold for carried interest payments with and without a catch-up clause.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
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</thead>
<tbody>
<tr>
<td>Remaining fund lifetime</td>
<td>$T_t - t$</td>
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<tr>
<td>Committed capital</td>
<td>$C$</td>
<td>Increases</td>
</tr>
<tr>
<td>Carried interest level</td>
<td>$c_{ci}$</td>
<td>Increases</td>
</tr>
<tr>
<td>Hurdle rate</td>
<td>$h$</td>
<td>Decreases</td>
</tr>
<tr>
<td>Speed of adjustment for the drawdown rate</td>
<td>$\kappa_1$</td>
<td>(Decreases)</td>
</tr>
<tr>
<td>Long-term drawdown rate</td>
<td>$\theta_1$</td>
<td>Increases</td>
</tr>
<tr>
<td>Volatility drawdown rate</td>
<td>$\sigma_1$</td>
<td>(Increases)</td>
</tr>
<tr>
<td>Speed of adjustment for the distribution rate</td>
<td>$\kappa_2$</td>
<td>(Increases)</td>
</tr>
<tr>
<td>Long-term distribution rate</td>
<td>$\theta_2$</td>
<td>Decreases</td>
</tr>
<tr>
<td>Volatility distribution rate</td>
<td>$\sigma_2$</td>
<td>(Decreases)</td>
</tr>
<tr>
<td>Expected return</td>
<td>$\mu$</td>
<td>Increases</td>
</tr>
<tr>
<td>Asset volatility</td>
<td>$\sigma$</td>
<td>Increases</td>
</tr>
<tr>
<td>Riskless rate</td>
<td>$r_f$</td>
<td>Decreases</td>
</tr>
</tbody>
</table>

5. **CONCLUSION**

This chapter demonstrates the application of a risk-neutral option-pricing approach to value private-equity fund fees. Our numerical analysis shows that the
performance related compensation accounts only for a relatively small portion of the total compensation of private equity fund managers. The framework also enables a detailed analysis of the determinants of private equity fund fee value. The results of this imply that a GP has various ways to influence the value of his future fee payments. For example, he can make riskier investments to increase the carried interest value. Thus, the fee structure of private equity funds gives rise to classical risk-shifting problems. However, the framework developed here does not take into account that such a risk-shifting behavior might also affect the ability of the GP to raise future funds. Undoubtedly, the ability for a GP to raise future funds will also be a very important consideration in his investment decisions. An extension of the model in this direction is a fruitful topic for future research.
In this appendix we present a numerical solution of valuation equation (15) using Monte Carlo simulation of a discrete-time approximation of the stochastic cash flow model presented in Section 2.2.

In order to implement the Monte Carlo simulation, we divide the time interval \([0, T]\) into \(K\) discrete intervals each of length \(\Delta t\). Then, we simulate all relevant quantities at equidistant time points \(t_k = k\Delta t\), where \(k = 1, \ldots, K\) and \(K = T/\Delta t\) holds.

Using these definitions, the dynamics of the capital drawdowns, (3), can be represented in discrete-time by

\[
\Delta D_{k+1} = \delta_{k+1} U_k \Delta t. \tag{20}
\]

Similarly, it holds that an appropriate discrete-time specification of the dynamics of the capital distributions, (6), is given by

\[
\Delta R_{k+1} = \rho_{k+1} V_k \Delta t, \quad \text{if} \quad k + 1 < K, \quad \text{and} \quad \Delta R_K = V_K. \tag{21}
\]

An appropriate scheme for approximating the risk-neutral dynamics of the drawdown and distribution rate, equations (16) and (17), is the Milstein scheme. Applying this scheme, it turns out

\[
\delta_{k+1} = \delta_k + \kappa_1 (\theta_1 - \delta_k) \Delta t + \sigma_1 \sqrt{\delta_k \Delta t} \varepsilon_{1,k+1} + \frac{1}{4} \sigma_1^2 \Delta t ((\varepsilon_{1,k+1})^2 - 1), \tag{22}
\]

\[
\rho_{k+1} = \rho_k + \kappa_2 (\theta_2 - \rho_k) \Delta t + \sigma_2 \sqrt{\rho_k \Delta t} \varepsilon_{2,k+1} + \frac{1}{4} \sigma_2^2 \Delta t ((\varepsilon_{2,k+1})^2 - 1), \tag{23}
\]

where \(\varepsilon_{1,1}, \varepsilon_{1,2}, \ldots, \varepsilon_{1,K}\) and \(\varepsilon_{2,1}, \varepsilon_{2,2}, \ldots, \varepsilon_{2,K}\) are i.i.d. sequences of standard normal variables that are assumed to be uncorrelated.

Finally, approximating the risk-neutral dynamics of the investment value, (18), using the Milstein scheme gives

\[
V_{k+1} = V_k [1 + (\mu + \lambda_3 \sigma) \Delta t] + \sigma V_k \varepsilon_{3,k+1} \sqrt{\Delta t} + \frac{1}{2} \sigma^2 V_k \Delta t ((\varepsilon_{3,k+1})^2 - 1) - \Delta R_{k+1} - \Delta D_{k+1}, \tag{24}
\]

where \(\varepsilon_{3,1}, \varepsilon_{3,2}, \ldots, \varepsilon_{3,K}\) is a third i.i.d. sequence of standard normal variables that is assumed to be uncorrelated with the sequences \(\varepsilon_{1,1}, \varepsilon_{1,2}, \ldots, \varepsilon_{1,K}\) and \(\varepsilon_{2,1}, \varepsilon_{2,2}, \ldots, \varepsilon_{2,K}\).

Using these discrete-time specifications, the risk-neutral carried interest dynamics (assuming no catch-up) can be approximated by

\[
\Delta CI_k = c_{cI} \max \{\Delta R_k - \Delta D_k - \Delta MF_k, 0\} 1(IRR_k > h), \tag{25}
\]

where \(IRR_k\) is a solution to

\[
\sum_{i=1}^{K} \frac{\Delta R_k - \Delta D_k - \Delta MF_k}{(1 + IRR_k)^i} = 0. \tag{26}
\]
To numerically solve equation (15), consider a Monte Carlo sampling experiment composed of \( M \) independent replications of the discrete-time approximations (20) to (26). Let \( \Delta R_{k,l} \ (\Delta D_{k,l}) \) denote the \( k \)th observation of the capital distributions (capital drawdowns) in the \( m \)th replication. Similarly, \( IRR_{k,l} \) denotes the \( k \)th observation of the internal rate of return in the \( m \)th replication. If the number of simulation iterations \( M \) is considerably large, the value of the outstanding carried interest with no catch-up in equation (15) can be approximated by computing

\[
V^C_{k} = \frac{1}{M} \sum_{l=1}^{M} \sum_{s=1}^{K} \left[ c_{cl} \max(\Delta R_{s,l} - \Delta D_{s,l} - \Delta MF_0, 0) 1[IRR_{s,l} > h] \right]. \tag{27}
\]

Note that the same approach can also be used to determine the value of the outstanding carried interest with a 100 percent catch-up provision. Throughout this chapter, we choose \( M = 100,000 \) simulation iterations to evaluate equation (27).
NOTES

1. For a more thorough introduction on the subject of private equity funds, for example, refer to Gompers and Lerner (1999b) or to the recent survey article of Phalippou (2007).

2. Besides management fee and the carried interest, private equity funds also often receive transaction and monitoring fees. Both types of additional fees are more common for buyout than for venture capital funds. A transaction fee is typically charged by buyout funds to their LPs when buying or selling a portfolio company. Additionally, many buyout funds charge monitoring fees to their portfolio companies. This is to compensate the fund for the time and effort spent by the GPs in advising and monitoring the portfolio companies. In most cases, these fees are shared with the LPs that receive 80 percent of the monitoring fees and GPs that receive 20 percent. Transaction and monitoring fees are not covered in the analysis of this paper and the interested reader is referred to Metrick and Yasuda (2010) for a more detailed overview.

3. As we model capital distributions and capital drawdowns separately, we have to restrict capital drawdowns to be strictly non-negative at any time \( t \) during the period \([0, T_c]\). The square-root diffusion was initially proposed by Cox et al. (1985) as a model of the short rate and is frequently denoted as CIR model. If \( \kappa_1, \theta_1 > 0 \), then \( \delta_t \) will never be negative. If \( 2\kappa_1\theta_1 \geq \sigma_1^2 \), then \( \delta_t \) remains strictly positive for all \( t \), almost surely. See Cox et al. (1985), p. 391.

4. Note here that under the specification (7), it will again hold that \( \rho_t \) will never be negative if \( \kappa_2, \theta_2 > 0 \) holds. If \( 2\kappa_2\theta_2 \geq \sigma_2^2 \), then \( \rho_t \) remains strictly positive for all \( t \), almost surely.

5. This follows as \( dD_t = 0 \) for \( t > T_c \), hold by definition.

6. Note that the internal rate of return is defined here by the continuously compounded rate of return that makes the net present value of the fund cash flows equal to zero.

7. Note that this relationship holds regardless of whether the expectation is defined under the objective measure \( \mathbb{P} \) or under the risk-neutral measure \( \mathbb{Q} \).

8. In this version of the ICAPM the expected return \( \mu_i \) of an asset \( i \) satisfies the relation \( \mu_i - r_f = \sigma_{iM} \), where \( \sigma_{iM} \) is the covariance of the return on asset \( i \) with the return of the market portfolio \( M \). This specification arises from the assumption of logarithmic utility. It permits to omit terms that are related to stochastic shifts in the investment opportunity set, which otherwise arise. See Merton (1973) and Brennan and Schwartz (1982) for a detailed discussion.
Note that this relationship holds regardless of whether the expectation is defined under the objective measure \( \mathbb{P} \) or under the risk-neutral measure \( \mathbb{Q} \).

9. For simplicity, we assume here that no management fees are paid to analyze the impact of the various model parameters on the value of the carried interest. Note that the management fees decrease the investment capital by the mechanism shown in equation (1) and therefore will also affect the carried interest payments.

10. For simplicity, we assume here that the commitment period \( T_c \) is equal to the legal lifetime of the fund \( T_l \), i.e., \( T_c = T_l \) holds.

11. With its second-order Taylor approximation, the Milstein scheme here provides a better approximation of the SDEs compared to the standard Euler scheme which is only based on a first-order Taylor approximation. For an arbitrary SDE, \( dX_t = \mu(X_t)dt + \sigma(X_t)dB_t \), the Milstein approximation takes the form

\[
\Delta X_{k+1} = \mu(X_k)\Delta t + \sigma(X_k)\sqrt{\Delta t} \varepsilon_{k+1} + \frac{1}{2} \sigma'(X_k)\Delta t[(\varepsilon_{k+1})^2 - 1],
\]

where \( \varepsilon_{k+1} \) is a standard normal variable and \( \sigma'(X_k) = d\sigma(X_k)/dX_k \) holds. For more details on how to approximate SDEs in discrete-time, see e.g. Glasserman (2003) or Kloeden and Platen (1999).

REFERENCES


